Std - XII | Maths-II (051) | Q. Paper Set III 1

Max. Marks 75

Time :- 3 hours

Q-I (A)

(1) Prove $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (2)

(2) Define (i) Bounded sequence. (2)

(ii) Limit of a sequence

(B) Find the following limit (Any two) (4)

(1) $\lim_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right)$, $m, n \in N$

(2) $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot 9^{r/n}$

(3) $\lim_{x \to \sqrt{2}} \frac{\pi - 4\sec^{-1}x}{x-\sqrt{2}}$

(C)

(1) Evaluvate $\lim_{n \to \infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right)$ (2)

(2) Answer any two (2)

(i) $\lim_{x \to 0} \frac{4\sin x - 6x}{7x - \tan x} = \_\_\_\_$

(ii) Express $\left\{ x \,\middle|\, \frac{1}{|2x-3|} \leq \frac{1}{5},\ x \in R - \left\{\frac{3}{2}\right\} \right\}$ as the complement of an interval

(iii) Find $\lim_{x \to \infty} x \left(\sqrt[x]{\frac{11}{12}} - 1 \right)$

(D)(1) Obtain the derivative of $e^{\sin^{-1}x}$ using the definition. (2)

(2) Fill up the Blanks by showing the necessary calculation (1)

$f(x) = |x| \Rightarrow f'(1) = \_\_\_$ and $f'(-1) = \_\_\_$

Q-II (A)

(1) State and prove the product Rule for differentiation. (2)

(2) a) Define: Derivative of a function at a point (1)

b) State Mean value theorem (1)

2

(B) Answer any two (4)

(1) If $y = \cos^{-1}x + \cos^{-1}\sqrt{1-x^2}$, $|x|<1$, find $\frac{dy}{dx}$

(2) If $x^p y^q = (x+y)^{p+q}$ find $\frac{dy}{dx}$

(3) If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ prove, $x^2y_2 + xy_1 + n^2y = 0$

(C) Answer any two. (4)

(1) Two vehicles start from the same point at the same point and move along lines perpendicular to each other. Their speeds are 45 kms/hr and 60 kms/hr respectively. Find the rate at which the distance between them increases 15 seconds after start.

(2) Prove x^x is minimum when $x = \frac{1}{e}$

(3) Determine whether Rolle's theorem can be applied to
$f(x) = 2x+3$, $x<3$
$= 15-2x$, $x \geq 3$, $x \in [1,5]$

(4) Length of each of 3 sides of a trapezium is 5a. What should be the length of the fourth side if its area is maximum possible?

[D] Answer the following questions. (3)

(1) Find the approximate value of $\log_{10} 9999$

(2) Prove that $e^x > 1+x$, $x \in R-\{0\}$

(3) Find the length of subnormal and subtangent at any point of the curve $y^2 = 4ax$

III (2)

(A)

(1) State and prove the method of substitution for indefinite integrals.

OR.

obtain formula of $\int \sqrt{a^2-x^2}\, dx$

(2) If f is continuous on $[0, 2a]$ prove that (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

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(2) Find the area of the region bounded by $y = 5x^2$ and $2x^2 - y + 9 = 0$

(3) The region bounded by $y = 4x - x^2$, $x = 1$, $x = 3$ and x-axis is divided into two parts of equal area by $x = c$. Find c.

[C] Solve any two. (4)

(1) A die is constructed in such a way that the probability of getting an integer on its face is proportional to that integer. A random variable X defined on sample space U associated with the random experiment of tossing a die is as follows.

$$X(u) = -2 \text{ if } u = 1, 2$$
$$= 4 \text{ if } u = 3, 4$$
$$= 8 \text{ if } u = 5, 6$$

obtain the probability ~~of~~ distribution of random variable X.

(2) Probability distribution of random variable X is given by $p(x) = P(X = x) = \frac{1}{M}$, $x = 1, 2, \ldots M$, where M is +ve integer. Find the mean and variance of random variable X. Also find $E(X(X-1))$.

(3) A random variable X follows binomial distribution whose mean and variance are $\frac{10}{3}$ and $\frac{10}{9}$ respectively. Find the parameters n and p of this binomial distribution. Also find $P(x > 0)$.

[D]

(1) Evaluate $\int \frac{dx}{x + 9x \log x}$ (1)

(2) (a) define : Homogeneous diff. eq^n.

(b) define : linear differential eq^n and explain the method to solve it.

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[B] Answer any two (4)

(1) $\int \frac{\sin x}{\sqrt{1+\sin x}}\,dx$

(2) $\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}}$

(3) $\int \frac{\sqrt{\cos x}}{\sin x}\,dx$

[C] Evaluate any two (4)

(1) Obtain $\int_1^e (2x+1)\,dx$ as limit of sum.

(2) $\int_0^a \frac{dx}{a+\sqrt{a^2-x^2}}$

(3) $\int_{-2}^{2} (x-3)\sqrt{4-x^2}\,dx$

[D]

(1) Evaluate $\int \frac{dx}{\sin^4 x+\cos^4 x}$ (2)

(2) Prove $\int e^x(f(x)+f'(x))\,dx = e^x f(x)+c$ (1)

IV

[A] Answer any two (4)

(1) Find the equation of family of curves with constant subtangent.

(2) Solve: $\frac{dy}{dx} = \sin(x+y)+\cos(x+y)$

(3) Solve: $y\,dx - x\,dy + \sqrt{x^2-y^2}\,dx = 0$

(4) A body having mass 60kg slides on the top of a table under a force of $54\sin 2t$ Newtons. Force of friction is 60 times its velocity and initially the velocity is zero. Express velocity is zero. Express velocity of the body as a function of time.

[B] Answer any two (4)

(1) If the region bounded by $x^2-y^2=a^2$ and $x=2a$ $(a>0)$ is rotated about y axis, find the volume of the solid of revolution.

(5)

V

[A] Prove the addition theorem of 2

(1) Probability for three events.

OR

Define (i) Axiomatic defn of probability 2

(ii) Prove $0 \leq P(A) \leq 1$.

(2) Prove that set function $P(A/B)$ is treated as a probability function of event A for a fixed event B, where $P(B) > 0$.

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[B] solve any two

(1) There are 4 white and 3 black balls in Box B1 and 4 black and 3 white balls in Box B_2. Two balanced coins are tossed. If coins show two heads Box B1 is selected and two balls are drawn at random from it, otherwise Box B_2 is selected and two balls are drawn at random from it. What is the probability that both selected balls are white?

(2) 4 letters are inserted at random one each in four covers. Find the probability that not more than one letter is in the proper cover.

(3) There are 4 red and 2 white distinct balls in a bag. These balls are arranged in a row. What is the probability that there is a white ball at both ends in a row? What is the probability that white balls are adjacent to each other in a row?

[C] 2

(1) Find out the probability of having 53 sundays in a randomly selected year.

(2) 1

(i) For two events A and B.

$P(A \cup B) = 0.9$ & $P(A \cap B') = 0.4$

$P(B \cap A') = 0.3$; Find the value of $P(A' \cup B')$.

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(ii) Prove for events A and B, $P(A \cap B) \geq P(A) + P(B) - 1$ (1)

[D]

(1) Distinguish between RAM and ROM. (1)

(2) Convert $(216.444)_8$ into binary and decimal forms. (2)

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(I)

Ans 1 (A)

(i) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Proof:- we know that $\sin\theta < \theta < \tan\theta$ for $0<\theta<\frac{\pi}{2}$

$\therefore 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$, $(\sin\theta \neq 0, \theta \neq 0)$

$\therefore \cos\theta < \frac{\sin\theta}{\theta} < 1$

Hence $\lim_{\theta \to 0^+} \cos\theta \leq \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \leq \lim_{\theta \to 0^+} 1$

This is because we have already noted if $f(x)<g(x)$ for all x in a deleted neighbourhood of a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Now $\lim_{x \to a} k = k$ implies $\lim_{\theta \to 0^+} 1 = 1$

and as cos is continuous, $\lim_{\theta \to 0^+} \cos\theta = \cos 0 = 1$

$1 \leq \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \leq 1$ which proves that $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

Next, suppose $\theta < 0$ put $\theta = -\alpha$ then $\theta \to 0-$ as $\alpha \to 0_+$

$\lim_{\theta \to 0-} \frac{\sin\theta}{\theta} = \lim_{\alpha \to 0+} \frac{\sin(-\alpha)}{-\alpha} = \lim_{\alpha \to 0+} \frac{\sin\alpha}{\alpha} = 1$

Hence $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Continuity over an interval

If a function $f: (a,b) \to R$ is continuous at every $x \in (a,b)$ we say that f is continuous on (a,b)

If f is defined on a closed interval $[a,b]$ and if

(1) f is continuous at each $x \in (a,b)$

(2) $\lim_{x \to a+} f(x) = f(a)$ [this result says that f is continuous from the right at a]

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(3) If for a sequence $\{a_n\}$, we can find a Positive number M such that $|a_n| \le M, \forall n \in N$, then the sequence $\{a_n\}$ is called to be bounded.

A-A 2) let $\{a(n)\}$ be any sequence and $f(n)$ be its n^{th} term. For given $\varepsilon > 0$ if there exists $m \in N$ such that $\forall\, n \geq m \Rightarrow |a_n - l| < \varepsilon$ then we say as $n \to \infty$ $a_n \to l$ is called limit of sequence

It is denoted by $\lim\limits_{n \to \infty} a_n = l$

[B]

(1) $\lim\limits_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right)$, $m, n \in N$

let $x - 1 = h$ As $x \to 1$, $h \to 0$.

$\therefore x = 1 + h$

$$= \lim_{x \to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n} \right)$$

$$= \lim_{h \to 0} \left(\frac{m}{1-(1+h)^m} - \frac{n}{1-(1+h)^n} \right)$$

$$= \lim_{h \to 0} \left(\frac{m}{1-(1+mh+m_{C_2}h^2 + \dots + h^m)} - \frac{n}{1-(1+nh+n_{C_2}h^2 + \dots + h^n)} \right)$$

$$= \lim_{h \to 0} \left[\frac{nmh + n\,m_{C_2}h^2 + \dots + nh^m - mnh - m\,n_{C_2}h^2 + \dots - mh^n}{} \right]$$

$$= \lim_{h \to 0} \frac{h^2 [n\,m_{C_2} + \dots + nh^{m-2} - m\,n_{C_2} + \dots - mh^{n-2}]}{h^2 (n + n_{C_2}h + \dots h^{n-1})(m + m_{C_2}h + \dots + h^{m-1})}$$

$$= \frac{n\,m_{C_2} - m\,n_{C_2}}{m \cdot n}, \quad (h \neq 0)$$

$$= \frac{\left[\frac{n(m)(m-1)}{1 \cdot 2} - \frac{m(n)(n-1)}{1 \cdot 2} \right]}{mn}$$

$$= \frac{nm[m-1-n+1]}{2m}$$

$$= \frac{m-n}{2}$$

(2) $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{1}{n} 9^{\frac{r}{n}}$

$$= \lim_{n\to\infty} \frac{1}{n}\left[9^{1/n} + 9^{2/n} + \ldots\ldots + 9^{n/n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\, \frac{9^{1/n}\left[(9^{1/n})^{n}-1\right]}{9^{1/n}-1} \qquad \left[s_n = \frac{a(1-r^n)}{(1-r)}\right]$$

(let $\frac{1}{n} = h$, As $n \to \infty$, $h \to 0$)

$$= \lim_{n\to 0} h\, \frac{9^{h} \times 8}{9^{h}-1}$$

$$= \lim_{h\to 0} \left[\frac{9^{h}\times 8}{\frac{9^{h}-1}{h}}\right]$$

$$= \frac{8}{\log_e 9} = 8 \log_9 e.$$

(3) $\lim\limits_{x\to\sqrt{2}} \frac{\pi - 4\sec^{-1}x}{x-\sqrt{2}}$

$$= \lim_{x\to\sqrt{2}} \frac{4\left(\frac{\pi}{4} - \sec^{-1}x\right)}{x-\sqrt{2}}$$

let $\sec^{-1}x = \theta$

$x = \sec\theta$ As $x \to \sqrt{2}$, $\theta \to \frac{\pi}{4}$

$$\therefore \lim_{\theta\to 4} \frac{4[\pi/4 - \theta]}{\frac{1}{\cos\theta} - \frac{1}{\cos\pi/4}}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{4\left(\frac{\pi}{4}-\theta\right)\cos\theta\cos\frac{\pi}{4}}{\cos\frac{\pi}{4} - \cos\theta}$$

$$= \lim_{\theta\to\frac{\pi}{4}} \frac{4\left(\frac{\pi/4-\theta}{2}\right)\cdot 2\cos\theta\,\cos\frac{\pi}{4}}{(-2)\sin\left(\frac{\pi/4-\theta}{2}\right)\times\sin\left(\frac{\pi/4+\theta}{2}\right)} = \lim_{\theta\to\frac{\pi}{4}} \frac{4\left(\frac{\frac{\pi}{4}-\theta}{2}\right)\cdot 2\cos\theta\cos\frac{\pi}{4}}{\sin\left(\frac{\frac{\pi}{4}-\theta}{2}\right)(-2)\sin\left(\frac{\frac{\pi}{4}+\theta}{2}\right)}$$

$$= \frac{4\cdot 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}{(-2)\,\frac{1}{\sqrt{2}}}$$

$$= \frac{-4 \cdot 2}{2\sqrt{2}} \times \frac{1}{\sqrt{2} \times \frac{1}{\sqrt{2}}}$$

$$= \frac{-2 \cdot 2}{\sqrt{2}}$$

$$= -2\sqrt{2}$$

[C]

(i) $\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right) = \sum_{r=1}^{n} \tan^{-1}\left(\frac{(r+1)-r}{1+r(r+1)}\right)$

$$= \sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1} r\right]$$

$$= \tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \tan^{-1}4 - \tan^{-1}3 + \tan^{-1}5 - \tan^{-1}4 \dots\dots \dots\dots + \tan^{-1}(n+1) - \tan^{-1}n$$

$$= \tan^{-1}(n+1) - \tan^{-1}1$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right) = \lim_{n\to\infty}\left[\tan^{-1}(n+1) - \tan^{-1}1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

(2)

(i) $\lim_{x\to 0} \frac{4\sin x - 6x}{7x - \tan x}$

$$= \frac{\lim_{x\to 0}\left(4\frac{\sin x}{x} - \frac{6x}{x}\right)}{\lim_{x\to 0}\left(\frac{7x}{x} - \frac{\tan x}{x}\right)}$$

$$= \frac{4-6}{7-1} \qquad , \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0}\frac{\tan\theta}{\theta}\right)$$

$$= -\frac{2}{6}$$

$$= -1/3$$

(11)

(2) $\frac{1}{|2x-3|} \le \frac{1}{5}$, $x \ne \frac{3}{2} \Leftrightarrow |2x-3| \geq 5$
$\Leftrightarrow |x-\frac{3}{2}| \geq \frac{5}{2}$

the compliment B of this set is $= \{x \,/\, |x-\frac{3}{2}| < \frac{5}{2}, x \in R\}$

so, $B = \left(\frac{3}{2}-\frac{5}{2}, \frac{3}{2}+\frac{5}{2}\right)$

$= (-1, 4)$

$\therefore \left\{x \,/\, \frac{1}{|2x-3|} \le \frac{1}{5}, x \in R - \{\frac{3}{2}\}\right\}$

$= R - (-1, 4)$

(3) $\lim_{x \to \infty} x\left(\left(\frac{\pi}{12}\right)^{1/x} - 1\right)$; let $\frac{1}{x} = h$

As $x \to \infty$, $h \to 0$

$= \lim_{h \to 0} \frac{\left(\frac{\pi}{12}\right)^h - 1}{h}$ $\quad \because \left(\lim_{h \to 0} \frac{a^h - 1}{h} = \log_e a\right)$

$= \log_e\left(\frac{\pi}{12}\right)$

[D] 1) let $f(x) = e^{\sin^{-1}x}$

let $\lim_{t \to x} \frac{f(t)-f(x)}{t-x}$

$= \lim_{t \to x} \frac{e^{\sin^{-1}t} - e^{\sin^{-1}x}}{t-x}$

$= \lim_{t \to x} \left[\frac{e^{\sin^{-1}x}\left[e^{\sin^{-1}t - \sin^{-1}x} - 1\right]}{\sin^{-1}t - \sin^{-1}x} \times \frac{\sin^{-1}t - \sin^{-1}x}{t-x}\right]$

$= e^{\sin^{-1}x} \times \lim_{t \to x} \frac{\sin^{-1}t - \sin^{-1}x}{t-x}$, $\left(\because \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right)$

let $\sin^{-1}t = a$ and $\sin^{-1}x = b$

As $t \to x$, $a \to b$

$= e^{\sin^{-1}x} \cdot \lim_{a \to b} \frac{a-b}{\sin a - \sin b}$

$= e^{\sin^{-1}x} \times \lim_{a \to b} \frac{a-b}{2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)}$

$= \frac{e^{\sin^{-1}x}}{\cos b} = \frac{e^{\sin^{-1}x}}{\sqrt{1-\sin^2 b}} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$\left(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right)$

$f(x) = |x|$. 12

(2) $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{|x|-|1|}{x-1}$

$= \lim_{x\to 1} \frac{x-1}{x-1} = 1$; $f'(1) = 1$

$\lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1} \frac{|x|-|-1|}{x-(-1)}$

$= \lim_{x\to -1} \frac{-x-1}{x-(-1)} = -1$; $f'(-1) = (-1)$

Q-II

(A)

Statement of the Product rule for differention.

If $f, g : (a, b) \to R$ are both differentiable at x, then fg is also differentiable at x and.

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{d(x)}f(x)$$

Proof:- $\lim_{t\to x} \frac{f(t)g(t) - f(x)g(x)}{t-x}$

$$= \lim_{t\to x} \frac{f(t)g(t) - f(t)g(x) + f(t)g(x) - f(x)g(x)}{t-x}$$

Now f, g are differentiable at x, $g(x)$ is constant and $\lim_{t\to x} f(t)$ exists. So by the working rules for limits,

$$\lim_{t\to x} \frac{f(t)g(t) - f(x)g(x)}{t-x} = \lim_{t\to x} f(t) \lim_{t\to x} \frac{g(t)-g(x)}{t-x} + g(x) \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$$

$$= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

as f is differentiable at x, so it is continuous at x

and so $\lim_{t\to x} f(t) = f(x)$

thus $(fg)' = fg' + gf'$, $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx} + g(x)\frac{d}{dx}f(x)$.

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(2) A

(2) (a)

Derivative of $f(x)$ at a point

let $f : (a, b) \to \mathbb{R}$, $t \in (a, b)$ be a function.

For a fixed x and for $t \neq x$.

$$\frac{f(t) - f(x)}{t - x}$$ is a function of the

$\therefore$ variable t, defined for all $t \in (a, b) - \{x\}$.

If $\lim\limits_{t \to x} \frac{f(t) - f(x)}{t - x}$ exists, it is called the

derivative of f at x.

(b) Mean Value theorem:

If f is continuous in $[a, b]$ and differentiable in (a, b) then we can always find $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

[B] (1)

$$y = \cos^{-1} x + \cos^{-1} \sqrt{1 - x^2}, \quad |x| < 1$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \frac{d}{dx}\left(\sqrt{1 - x^2}\right)$$

$$= \frac{-1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - (1 - x^2)}} \times \frac{1}{2\sqrt{1 - x^2}} \frac{d}{dx}(1 - x^2)$$

$$= \frac{-1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - 1 + x^2}} \times \frac{1}{2\sqrt{1 - x^2}} (0 - 2x)$$

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$$= \frac{-1}{\sqrt{1-x^2}} + \frac{x}{|x|\sqrt{1-x^2}}.$$

If $x > 0$, then $|x| = x$,

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} + \frac{x}{x\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= 0$$

If $x < 0$, then $|x| = -x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} + \frac{x}{-x\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-2}{\sqrt{1-x^2}}$$

At $x = 0$. It is not differentiable.

(2) $$x^p y^q = (x+y)^{p+q}$$

Take log on both sides w.r.to base e

$$\therefore \log_e(x^p y^q) = \log_e(x+y)^{p+q}$$

$$\therefore p\cdot\log_e x + q\log_e y = (p+q)\log(x+y)$$

diff w.r.to x

$$\therefore \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y}\left[1 + \frac{dy}{dx}\right]$$

(15)

$\therefore \frac{dy}{dx}\left[\frac{q}{y} - \frac{(p+q)}{(x+y)}\right] = \frac{p+q}{x+y} - \frac{p}{x}$

$\therefore \frac{dy}{dx}\left[\frac{qx+qy-py-qy}{y(x+y)}\right] = \frac{px+qx-px-py}{x(x+y)}$

$\frac{dy}{dx}\frac{(qx-py)}{y(x+y)} = \frac{(qx-py)}{x(x+y)}$

$\therefore \frac{dy}{dx} = \frac{y}{x}.$

(3) $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n = n[\log x - \log n]$

differentiating w.r. to x,

$\therefore \frac{-1}{\sqrt{1-\frac{y^2}{b^2}}} \times \frac{1}{b}\cdot\frac{dy}{dx} = \frac{n}{x}$

$\therefore \frac{-1}{\sqrt{b^2-y^2}} \times y_1 = \frac{n}{x}$

$\therefore -xy_1 = n\sqrt{b^2-y^2})$

squaring.

$x^2y_1^2 = n^2(b^2-y^2)$

$\therefore x^2 2y_1y_2 + y_1^2 2x = -n^2 2yy_1$

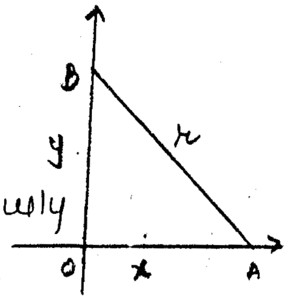
$\therefore x^2 2y_1y_2 + 2xy_1^2 + 2yy_1n^2 = 0$

cancelling $2y_1$

$\therefore x^2y_2 + xy_1 + n^2y = 0.$

[c]

r(1) suppose that vehicles start from O suppose that they are at A and B after 15 seconds. their speeds are 45 kms/hr and 60 kms/hr respectively



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$OA = x =$ distance travelled ; $\frac{dx}{dt} = 45$ km/hr

$= 45 \times \frac{15}{3600}$

$= \frac{3}{16}$

$OB = y = 60 \times \frac{15}{3600}$; $\frac{dy}{dt} = 60$ km/hr

$= \frac{15}{60} = \frac{1}{4} = \frac{4}{16}$

In ΔOAB $m\angle O = 90°$

$AB^2 = OA^2 + OB^2$

$\therefore r^2 = x^2 + y^2$ $\therefore r^2 = \left(\frac{3}{16}\right)^2 + \left(\frac{4}{16}\right)^2$

Differentiating w.r.t. t $= \left(\frac{5}{16}\right)^2$

$\cancel{2}r \frac{dr}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$ $\therefore r = \frac{5}{16}$

$\therefore \frac{5}{16} \frac{dr}{dt} = \frac{3}{16} \times 45 + \frac{4}{16} \times 60$

$\therefore \frac{dr}{dt} = 27 + 48$

$= 75$ kms/hr

(2) let $y = x^x$

$\therefore \log y = x \log x$

$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x$

$\therefore \frac{dy}{dx} = x^x(1 + \log x)$

For maxima and minima $\frac{dy}{dx} = 0$.

$\Rightarrow x^x(1 + \log x) = 0$

$\therefore 1 + \log x = 0$

$\therefore \log_e x = -1$

$\therefore x = e^{-1}$

$= \frac{1}{e}$

(17)

Now $\frac{d^2y}{dx^2} = \frac{x^x}{x} + (1+\log x)\frac{d}{dx}(x^x)$

$= x^{x-1} + x^x(1+\log x)^2$

At $x = \frac{1}{e}$, $\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{1/e - 1}$

$= e\left(\frac{1}{e}\right)^{1/e} > 0$.

$\therefore$ For $x = \frac{1}{e}$, x^x has minimum value.

(3) $f(x) = 2x+3$, $x < 3$

$= 15-2x$, $x \geq 3$.

continuity at $x = 3$

$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} (5-2x) = 9$.

$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (2x+3) = 9$

$f(3) = 15-6 = 9$

$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} f(x) = f(3) = 9$

$\therefore$ Function is continuous at $x = 3$

Differentiability

$\lim_{x \to 3+} \frac{f(x)-f(3)}{x-3} = \lim_{x \to 3+} \frac{15-2x-9}{x-3}$

$= \lim_{x \to 3+} \frac{-2(x-3)}{(x-3)}$, As $x \to 3^+$, $x \in (3, 3+\delta)$

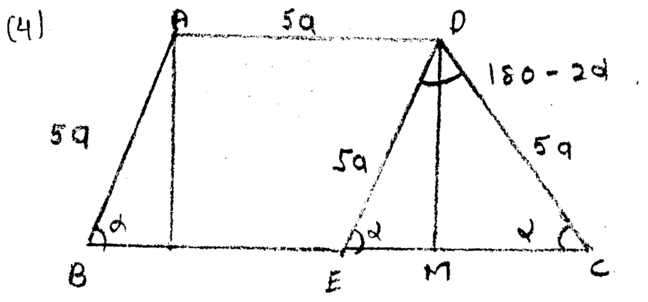
$= -2$

$\lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$ does not exist

$\therefore$ f is not differentiable at $x=3$

$\therefore$ Roller's theorem cannot be applied.

(4)



Suppose $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$ and $E \in \overline{BC}$

let $m\angle ABE = \alpha$

then $m\angle DEC = \alpha$

$\therefore m\angle EDC = 180 - 2\alpha$

Area of trapezium ABCD

$f(\alpha) =$ Area of $\square$gm ABED + Area of Δ DEC

$f(\alpha) = 5a \times 5a \sin\alpha + \frac{1}{2} 5a \cdot 5a \sin(180 - 2\alpha)$

$= \frac{25a^2}{2} [2\sin\alpha + \sin 2\alpha]$

For maxima or minima $f'(\alpha) = 0$

$\Rightarrow 2\cos\alpha + 2\cos 2\alpha = 0$

$\Rightarrow 2\cos\alpha + 2(2\cos^2\alpha - 1) = 0$

$\Rightarrow 2\cos^2\alpha + \cos\alpha - 1 = 0$

$\Rightarrow (2\cos\alpha - 1)(\cos\alpha + 1) = 0$

But $\cos\alpha = -1$ } is not possible, therefore $\cos\alpha = \frac{1}{2}$

$\Rightarrow \alpha = \pi$

Now $f''(\alpha) = \frac{25a^2}{2} (-2\sin\alpha - 4\sin 2\alpha)$

< 0

$\therefore f$ has local minimum when $\cos\alpha = \frac{1}{2} \therefore \alpha = \frac{\pi}{3}$

If $\alpha = \frac{\pi}{3}$ $EC = 2EM$

$= 2 \times 5a\cos\alpha$

$= 10 \times \frac{1}{2}$

$= 5a.$

(19)

$BC = BE + EC$
$= 5a + 5a$
$= 10a$

[D]

(1) $f(x) = \log_{10} x$ $\qquad \log_{10} e = 0.4343$.

$f(x+h) \cong f(x) + h f'(x)$ $\qquad f'(x) = \dfrac{\log_{10} e}{x}$

$$\log_{10} 9999 = f(9999)$$
$$= f(10^4 - 1)$$
$$\cong f(10^4) - 1 f'(10^4)$$
$$\cong 4 - \frac{0.4343}{10^4}$$
$$\cong 4 - 0.00004343$$
$$\cong 3.99995657$$

(2) let $f(x) = e^x - 1 - x$, $x \in R - \{0\}$

$f(x)$ is continuous and differentiable on R

when $x > 0$, $f'(x) = e^x - 1$
> 0 $\quad (\because e > 1 \Rightarrow e^x > 1)$

$\Rightarrow$, f is increasing

$\therefore x > 0 \Rightarrow f(x) > f(0)$
$\therefore e^x - 1 - x > 0$
$\therefore e^x - 1 > x$ $\quad$ (ie) $e^x > 1 + x$

when $x < 0$, $f'(x) = e^x - 1 < 0$
$\Rightarrow$ f is Decreasing.
$\therefore x < 0 \Rightarrow f(x) > f(0)$
$\Rightarrow e^x - 1 - x > 0$
$\Rightarrow e^x > 1 + x$.

$\therefore e^x > 1 + x$, $x \in R - \{0\}$.

(3) $y^2 = 4ac$

$\therefore 2y \dfrac{dy}{dx} = 4a$

$\therefore \dfrac{dy}{dx} = \dfrac{2a}{y}$

At a point (x, y) on the curve,

length of subnormal $= \left| y \dfrac{dy}{dx} \right| = \left| y \cdot \dfrac{2a}{y} \right| = |2a|$
$= \text{constant}$

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length of subtangent $= \left|\frac{y}{\frac{dy}{dx}}\right| = \left|\frac{y}{\frac{2a}{y}}\right| = \left|\frac{y^2}{2a}\right| = \left|\frac{4ax}{2a}\right|$

$= |2x|$

III

[A] 1) theory

2) theory

[B]

1) $\int \frac{\sin x}{\sqrt{1+\sin x}}\, dx$

$= \int \frac{\cos(\pi/2 - x)}{\sqrt{1+\cos(\pi/2 - x)}}\, dx$

$= \int \frac{2\cos^2(\pi/4 - x/2) - 1}{\sqrt{2\cos^2(\pi/4 - \frac{x}{2})}}$

$= \sqrt{2}\int \left[\cos\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{1}{2}\sec\left(\pi/4 - \frac{x}{2}\right)\right] dx$

$= \sqrt{2}\left[\frac{\sin(\pi/4 - x/2)}{-1/2} - \frac{1}{2}\frac{\log|\sec(\pi/4 - x/2) + \tan(\pi/4 - x/2)|}{-1/2}\right] + c$

$= -2\sqrt{2}\left[\sin\left(\frac{\pi}{4} - \frac{x}{2}\right) - \frac{1}{2}\log\left|\sec\left(\frac{\pi}{4} - \frac{x}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right|\right] + c$

(2) $\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}}$

let $x^{1/12} = t$

$x = t^{12}$

$\frac{dx}{dt} = 12t^{11}$

$\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} = 12\int \frac{t^{11}\, dt}{t^4 + t^3}$

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$$= 12\int \frac{t^8}{t+1}\,dt$$

$$= 12\int \frac{(t^8+1)-1}{(t+1)}\,dt$$

$$= 12\int \frac{t^7-t^6+t^5-t^4+t^3-t^2+t-1\,(t+1)}{(t+1)} - \int \frac{dt}{t+1}$$

$$= 12\left[\int [t^7-t^6+t^5-t^4+t^3-t^2+t-1]\,dt - \int \frac{dt}{t+1}\right]$$

$$= 12\left[\frac{t^8}{8}-\frac{t^7}{7}+\frac{t^6}{6}-\frac{t^5}{5}+\frac{t^4}{4}-\frac{t^3}{3}+\frac{t^2}{2}-t - \log(t+1)\right]+c$$

$$= 12\left[\frac{x^{2/3}}{8}-\frac{x^{7/12}}{7}+\frac{x^{1/2}}{6}-\frac{x^{5/12}}{5}+\frac{x^{1/3}}{4}-\frac{x^{1/4}}{3}+\frac{x^{1/6}}{2}-x^{1/12}\right.$$

$$\left.-\log|x^{1/12}+1|\right]+c$$

(3) $\int \frac{\sqrt{\cos x}}{\sin x}\,dx$

let $\sqrt{\cos x} = t$

$\therefore \cos x = t^2$

$\therefore -\sin x\,dx = 2t\,dt$

$$\therefore \int \frac{\sqrt{\cos x}}{\sin x}\,dx = \int t\,\frac{(-2t)\,dt}{\sin^2 x} = -\int \frac{2t^2}{1-t^4}\,dt$$

$$= \int \frac{2t^2}{t^4-1}\,dt$$

$$= \int \frac{(t^2-1)+(t^2+1)}{t^4-1}\,dt$$

$$= \int \frac{t^2-1}{t^4-1}\,dt + \int \frac{dt}{t^2-1}$$

$$= \tan^{-1} t + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

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$$= \tan^{-1}\sqrt{\cos x} + \frac{1}{2}\log\left|\frac{\sqrt{\cos x}-1}{\sqrt{\cos x}+1}\right| + c$$

(c)

(1) $\int_1^e (2x+1)\,dx$

Here $f(x) = 2x+1$ is continuous on $[1,e]$ Dividing $[1,e]$ into n subintervals of equal length h

Here $a=1$, $b=e$

$h = \frac{b-a}{n} = \frac{e-1}{n}$

$f(a+ih) = f(1+ih) = 2(1+ih)+1$
$= 3+2ih$

$$\int_1^e (2x+1)\,dx = \lim_{h\to 0} h\sum_{i=1}^{n} f(a+ih)$$

$$= \lim_{h\to 0} h\sum_{i=1}^{n}(3+2ih)$$

$$= \lim_{h\to 0}\left[h\sum_{i=1}^{n}3 + 2h^2\sum_{i=1}^{n} i\right]$$

$$= \lim_{h\to 0}\left[h\cdot 3n + 2h^2\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(e-1)}{n}\times 3n + 2\frac{(e-1)^2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[3(e-1) + (e-1)^2\left(1+\frac{1}{n}\right)\right]$$

$= 3(e-1) + e^2 - 2e + 1$

$= 3e - 3 + e^2 - 2e + 1$

$= e^2 + e - 2$

(2) $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

let $x = a\sin\theta$ $\quad \theta \in (0, \frac{\pi}{2})$

$dx = a\cos\theta\, d\theta$

when $x=0$, $a\sin\theta = 0 \Rightarrow \theta = c$

when $x=a$, $a\sin\theta = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta + \sqrt{a^2 - a^2\sin^2\theta}}$$

$$I = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta + \cos\theta} \quad \text{—— (1)}$$

$$I = \int_0^{\pi/2} \frac{\cos(\pi/4 - \theta)\, d\theta}{\sin(\pi/2 - \theta) + \cos(\pi/2 - \theta)}$$

$$= \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta + \sin\theta} \quad \text{—— (2)}$$

Adding (1) and (2)

$$2I = \int_0^{\pi/2} d\theta$$

$$= [\theta]^{\pi/2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

(3) $\displaystyle\int_{-2}^{-2} (x-3)\sqrt{4-x^2}\, dx$

$$I = \int_{-2}^{2} (x-3)\sqrt{4-x^2}\, dx$$

$$= \int_{-2}^{2} x\sqrt{4-x^2}\, dx - 3\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

let $f(x) = x\sqrt{4-x^2}$

$f(-x) = -x\sqrt{4-(-x)^2}$

$= -x\sqrt{4-x^2}$

$= -f(x)$

$\Rightarrow f$ is odd function.

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let $g(x) = \sqrt{4-x^2}$

$g(-x) = \sqrt{4-(-x)^2}$

$= \sqrt{4-x^2}$

$= g(x)$

$\Rightarrow$ g is even function

Using theorem $\int_{-a}^{a} f(x)\,dx = 0$ if f is an odd function

we have $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$.

Using theorem $\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx$ if g(x) is an even function

we have $\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\int_{0}^{2} \sqrt{4-x^2}\,dx$

$\therefore I = 0 - 6\int_{0}^{2} \sqrt{4-x^2}\,dx$

$= -6\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2$

$= -6\left[0 + 2\cdot\frac{\pi}{2} - 0 - 0 \right]$

$= -6\pi$

[D]

(1) $I = \int \frac{dx}{\sin^4 x + \cos^4 x}$

$= \int \frac{dx}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}$

$= \int \frac{dx}{1 - \frac{1}{4} + \frac{1}{4}\cos 4x} = \int \frac{dx}{\frac{3}{4} + \frac{1}{4}\cos 4x}$

$= 4\int \frac{dx}{3 + \cos 4x}$

let $\tan 2x = t$

$\therefore 2\sec^2 2x\,dx = dt$

$\therefore dx = \frac{dt}{2(1+t^2)}$

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$$I = 4\int \frac{dt}{2(1+t^2)\left(3+\frac{1-t^2}{1+t^2}\right)}$$

$$= 4\int \frac{dt}{2(1+t^2)\frac{(3+3t^2+1-t^2)}{(1+t^2)}}$$

$$= 2\int \frac{dt}{2t^2+4}$$

$$= \int \frac{dt}{t^2+2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan 2x}{\sqrt{2}}\right)+c$$

(2) theory.

[V]

[A]

(1) let the constant subtangent be a

$$\left|\frac{y}{\frac{dy}{dx}}\right| = a$$

$$\therefore \frac{\frac{dy}{dx}}{y} = \frac{1}{a} = k$$

$$\therefore \frac{dy}{dx} = ky$$

$\therefore \int \frac{dy}{y} = \int k\,dx$ (variables separable Method)

$$\therefore \log y = kx + \log c$$

$$\therefore \log_e\left(\frac{y}{c}\right) = kx$$

$$\therefore \frac{y}{c} = e^{kx}$$

$\therefore y = ce^{kx}$ where c is constant

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(2) Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

let $x+y = t$

$\therefore \quad 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \quad \frac{dt}{dx} - 1 = \sin t + \cos t$

$\therefore \quad \frac{dt}{dx} = 1 + \sin t + \cos t$, (variables seperable form)

$\therefore \int \frac{dt}{1+\sin t + \cos t} = \int dx$

$\therefore \int \frac{dt}{1+\cos t + \sin t} = \int dx$

$\therefore \int \frac{dt}{2\cos^2\frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2}} = x + c.$

$\therefore \log\left|1+\tan\frac{t}{2}\right| = x + c$

$\therefore \quad 1 + \tan\frac{t}{2} = e^{x+c}$

$= e^x . e^c$

$\therefore \quad 1 + \tan\frac{(x+y)}{2} = Ke^x$ where $e^c = k$.

(3) $y\,dx - x\,dy + \sqrt{x^2-y^2}\,dx = 0$.

$\therefore \quad y - x\frac{dy}{dx} = -\sqrt{x^2-y^2}$

$\therefore \quad x\frac{dy}{dx} = y + \sqrt{x^2-y^2}$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \sqrt{1-(y/x)^2}$

This is Homogeneous differentiable equation,

let $y = vx$

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = v + \sqrt{1-v^2}.$

$\therefore \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{dx}{x}$

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$\therefore \sin^{-1} v = \log x + \log c$.

$\therefore \sin^{-1}\left(\frac{y}{x}\right) = \log cx$, is general solution.

(4) let v be the velocity at time t

Force = mass x acceleration.

Force applied = 54 sin 2t

Force of friction = 60v

$\therefore 60 \frac{dv}{dt} = 52 \sin 2t - 60v$

$\therefore \frac{dv}{dt} + v = \frac{54}{60} \sin 2t$ is a linear

diff eq^n

Integrating factor $= e^{\int p\, dt} = e^{\int dt} = e^t$

$\therefore e^t \frac{dv}{dt} + e^t v = \frac{9}{10} e^t \sin 2t$

$\therefore \frac{d}{dt}(e^t v) = \frac{9}{10} e^t \sin 2t$

$\therefore v e^t = \frac{9}{10} \frac{e^t}{(1+4)} (\sin 2t - 2\cos 2t) + c$

$\therefore v = \frac{9}{50}(\sin 2t - 2\cos 2t) + c e^{-t}$

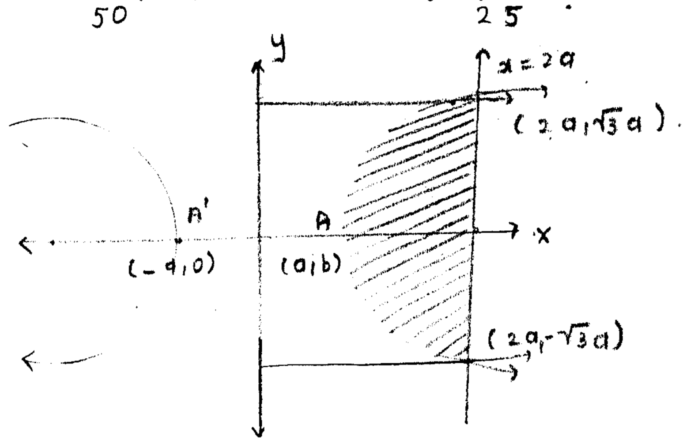
when $t = 0$, $v = 0$

$0 = \frac{9}{50}(0 - 2\cos 0) + c$

$c = \frac{9}{25}$

$v = \frac{9}{50}(\sin 2t - 2\cos 2t) + \frac{9}{25} e^{-t}$.

[B] [B]



S- 28

$x = 2a$, $x^2 - y^2 = a^2$ intersect

Point of intersection

$4a^2 - y^2 = a^2$

$\Rightarrow y^2 = 3a^2$

$\Rightarrow y = \pm\sqrt{3}a$

Points are $(2a, -\sqrt{3}a)$ and $(2a, \sqrt{3}a)$

If the region bounded by the curve $x^2 - y^2 = a^2$ $y = -\sqrt{3}a$, $y = \sqrt{3}a$ and y axis is revolved around y axis, then the volume of the solid generated.

$$V = \pi \int_{-\sqrt{3}a}^{\sqrt{3}a} \left[[f_1(y)]^2 - (f_2(y))^2 \right] dy \quad \text{where}$$

$f_1(y) = 2a$ and $f_2(y) = \sqrt{a^2 + y^2}$

$$V = \pi \int_{-\sqrt{3}a}^{\sqrt{3}a} \left(4a^2 - (a^2 + y^2)\right) dy$$

$$= 2\pi \int_0^{\sqrt{3}a} (3a^2 - y^2)\, dy, \quad (\because \text{even function})$$

$$= 2\pi \left[3a^2 y - \frac{y^3}{3} \right]_0^{\sqrt{3}a}$$

$$= 2\pi \left[3a^2\sqrt{3}a - \frac{3\sqrt{3}a^3}{3} - 0 \right]$$

$$= 2\pi [3\sqrt{3}a^3 - \sqrt{3}a^3]$$

$$= 4\sqrt{3}\pi a^3 \text{ units}$$

(2) Point of intersection of the curves $y = 5x^2$, and $2x^2 - y + 9 = 0$

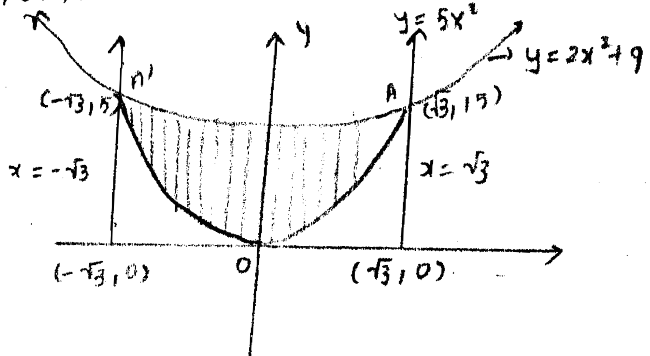
$5x^2 = 2x^2 + 9$

$\therefore 3x^2 = 9$

$\therefore x^2 = 3$

$\therefore x = \pm\sqrt{3}$

Po point of x^y are $(\sqrt{3}, 15)$ and $(-\sqrt{3}, 15)$ 29



Required area = $|I|$ where

$$I = \int_a^b [f_1(x) - f_2(x)]\,dx \quad \text{where } f_1(x) = 2x^2 + 9$$

and

$$f_2(x) = 5x^2$$

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} (2x^2 + 9 - 5x^2)\,dx$$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)\,dx$$

$$= 2\left[9x - 3\frac{x^3}{3}\right]_0^{\sqrt{3}}$$

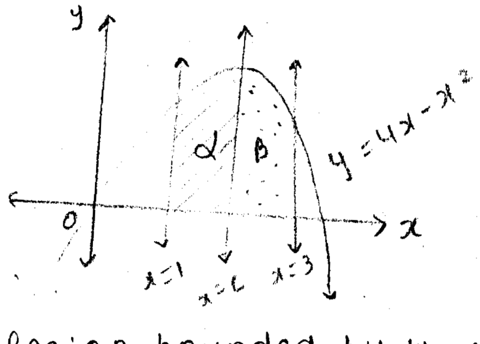
$$= 2[9\sqrt{3} - 3\sqrt{3}]$$

$$= 12\sqrt{3}$$

$A = |I|$

$= |12\sqrt{3}|$

$= 12\sqrt{3}$ units.



Region bounded by the curve $y = 4x - x^2$, x axis the lines $x = 1$ and $x = 3$ is divided into two equal parts α and β by the line $x = c$

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Here $\alpha = \beta$

ie) $\int_1^c y\,dx = \int_c^3 y\,dx$.

$\therefore \int_1^c (4x - x^2)\,dx = \int_c^3 (4x - x^2)\,dx$

$\therefore \left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_1^c = \left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_c^3$

$\therefore \left[2x^2 - \frac{x^3}{3}\right]_1^c = \left[2x^2 - \frac{x^3}{3}\right]_c^3$

$\therefore \left(2c^2 - \frac{c^3}{3}\right) - \left(2 - \frac{1}{3}\right) = (18 - 9) - \left(2c^2 - \frac{c^3}{3}\right)$

$\therefore 4c^2 - \frac{2c^3}{3} = 9 + 2 - \frac{1}{3}$

$\therefore 12c^2 - 2c^3 = 33 - 1$

$\therefore 12c^2 - 2c^3 = 32$ (ie) $\Rightarrow 6c^2 - c^3 = 16$

$\therefore c^3 - 6c^2 + 16 = 0$.

$c = 2$ is a factor

$c^3 - 6c^2 + 16 = 0$.

$\therefore (c-2)(c^2 - 4c - 8) = 0$

$\therefore c = 2$, or $c^2 - 4c - 8 = 0$

$\therefore c = 2$ or $c = \frac{4 \pm \sqrt{16+32}}{2}$

$= \frac{4 \pm 4\sqrt{3}}{2}$

$= 2 \pm 2\sqrt{3}$

$2 \pm 2\sqrt{3} \notin [1, 3]$ is not possible

$\therefore c = 2$

$\boxed{\text{IV}}$

[c] $P(\{i\}) \propto i$

$P(\{i\}) = ki \quad i = 1 \ldots 6$

$P(\{1\}) = k$, $P(\{3\}) = 3k$, $P(\{6\}) = 6k$,

$P(\{2\}) = 2k$, $P(\{4\}) = 4k$,

$P(\{5\}) = 5k$

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These events are mutually exclusive

$$\sum_{i=1}^{6} P(\{i\}) = 1$$

$k+2k+3k+4k+5k+6k = 1$

$\therefore \quad 21k = 1$

$\therefore \quad k = \frac{1}{21}$

$P(\{1\}) = \frac{1}{21}$, $P(\{2\}) = \frac{2}{21}$, $P(\{3\}) = \frac{3}{21}$, $P(\{4\}) = \frac{4}{21}$

$P(\{5\}) = \frac{5}{21}$ $P(\{6\}) = \frac{6}{21}$

Element of U	$P(\{U\})$	$X(U) = x$	$P(x) = P(X(U) = x)$
1 2	$\frac{1}{21}$ $\frac{2}{21}$	-2	$\frac{1}{21} + \frac{2}{21} = \frac{3}{21}$
3 4	$\frac{3}{21}$ $\frac{4}{21}$	4	$\frac{3}{21} + \frac{4}{21} = \frac{7}{21}$
5 6	$\frac{5}{21}$ $\frac{6}{21}$	8	$\frac{5}{21} + \frac{6}{21} = \frac{11}{21}$

$X = x$	-2	4	8
$P(x)$	$\frac{3}{21}$	$\frac{7}{21}$	$\frac{11}{21}$

(2) $P(x) = P(X = x) = \frac{1}{M}$, $x = 1, \ldots M$

x	1	2	3	....	M
$P(x)$	$\frac{1}{M}$	$\frac{1}{M}$	$\frac{1}{M}$	....	$\frac{1}{M}$

$$E(X) = \sum_{i=1}^{M} x_i \, p(x_i)$$

$$E(X^2) = \sum_{i=1}^{M} x_i^2 \, p(x_i)$$

$$= \frac{1}{M} \sum_{i=1}^{M} i^2 = \frac{M(M+1)(2M+1)}{6 * M} = \frac{(M+1)(2M+1)}{6}$$

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$$= \frac{(M+1)(2M+1)}{6}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{(M+1)(2M+1)}{6} - \left(\frac{M+1}{2}\right)^2$$

$$= \frac{(M+1)}{12}\left[2(2M+1) - 3(M+1)\right]$$

$$= \frac{M+1}{12}(M-1) = \frac{M^2-1}{12}$$

$$\Rightarrow \sigma_x^2 = \frac{M^2-1}{12}.$$

$$E(x(x-1)) = E(x^2-x)$$

$$= E(x^2) - E(x)$$

$$= \frac{(M+1)(2M+1)}{6} - \frac{(M+1)}{2}$$

$$= \frac{M+1}{2}\left[\frac{2M+1}{3} - 1\right]$$

$$= \frac{(M+1)}{2}\frac{(2M-2)}{3} = \frac{M^2-1}{3}.$$

For a Binomial distribution,

$$\text{Mean} = np = \frac{10}{3}$$

$$\text{Variance} = npq = \frac{10}{9}$$

$$q = \frac{npq}{np} = \frac{10 \times 3}{9 \times 10} = \frac{1}{3}$$

$$p = 1-p = 1-\frac{1}{3} = \frac{2}{3}$$

$$np = \frac{10}{3} \Rightarrow n = \frac{10}{3} \times \frac{3}{2} = 5$$

$$P(x>0) = 1 - P(x=0)$$

$$= 1 - 5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5$$

$$= 1 - \frac{1}{3^5} = 1 - \frac{1}{243} = \frac{242}{243}$$

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[D]

(1) $\int \frac{dx}{x+9x\log x}$

$= \int \frac{dx}{x(1+9\log x)}$

let $\log x = t$ $\quad = \int \frac{dt}{1+9t}$

$= \frac{1}{9}\log|1+9t| + C$

$\therefore = \frac{1}{9}\log|1+9\log x| + C$

(2) a theory

b theory

V [A] 1) Theory

2) theory

[B] (1) sample space for tossing two coins

$= \{HH, HT, TH, TT\}$

let B_1 be the event of choosing Box B_1

$\therefore P(B_1) = \frac{1}{4}$

let B_2 be the event of chosing Box B_2 -

$P(B_2) = \frac{3}{4}$

let A be the event of selecting 2 white balls.

$P(A) = P(B_1)P(A|B_1) + P(B_2).P(A|B_2)$

$= \frac{1}{4} \times \frac{4C_2}{7C_2} + \frac{3}{4} \times \frac{3C_2}{7C_2}$

$= \frac{1}{4} \times \frac{4\times3}{7\times6} + \frac{3}{4} \times \frac{3\times2}{7\times6}$

$= \frac{1}{14} + \frac{3}{28} = \frac{5}{28}$

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(2) let the letters be numbered 1, 2, 3 and the cover be 1, 2, 3 & 4 so that (i) $i = 1, 2, 3, 4$ is the correct arrangement.

let A devote the event that more than one letter is put in the proper cover

consider the compliment of A ie A'

(i) 2 letters are put in the proper cover
(ii) 3 letters are put in the proper cover
(iii) 4 letters are put in the proper cover

(i) 2 letters are put in the proper cover in 6 ways

Cover $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ correct allignment

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & 3 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 1 & 4 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 3 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 2 & 4 \\ \hline\end{array}$

$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 3 & 4 \\ \hline\end{array}$

(ii) If 3 letters are in proper cover, then the fourth automatically in the proper cover → No possiblity

(iii) Four letters are put in the proper cover only in one way

Arrangement of 4 letters in 4 covers can be done in one way.

Arrangement of 4 letters in 4 covers can be done in $^4P_4 = 4 \times 3 \times 2 \times 1 = 24$ ways.

Total number of ways of putting more than one letter in proper cover will be $= 6 + 0 + 1$
$= 7$ ways

$P(A') = \frac{7}{24}$

$P(A) + P(A') = 1$; $P(A) = 1 - P(A')$
$= 1 - \frac{7}{24} = \frac{17}{24}$

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(3) let us denote the white balls by W_1 and red balls by R_1, R_2, R_3, R_4

6 balls can arranged in 6 places in $6!$ ways.

No. of elements in sample space $= 6!$
$= 720$

(i) $W_1, R_1, R_2, R_3, R_4, W_2$ or $W_2\ R_1\ R_2\ R_3\ R_4\ W_1$

let A denote the event that the first and last ball in a row is white.

Arrangement of two balls in the first and last can be done in 2 ways and the 4 red balls in between can be arranged in 4! ways.

$$P(A) = \frac{2 \times 4!}{6!} = \frac{2 \times 24}{720}$$

$$= \frac{1}{15}$$

(ii) Consider the event B in which two white balls are adjacent to each other

consider the balls B_1, B_2 as one unit

$B_1 B_2 R_1\ R_2 R_3\ R_4$

5 balls in 5 places can be arranged in 5! ways among themselves, can be arranged in 2 ways

$$\text{so, } P(B) = \frac{5! \times 2!}{6!} = \frac{5! \times 2}{6 \times 5!} = \frac{1}{3}$$

[c] let A denote the event that selected year is not a leap year (ie) 365 days

let B denote the event that selected year is a leap year (ie) 366 days

$P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{4}$

let C denote the event of having 53 sundays.

$P(C) = P(A)\,P(C/A) + P(B)\,P(C/B)$

For $P(C/A) \Rightarrow$ Not a leap year

$\therefore$ No of days = 365.

$\therefore \frac{365}{7} =$ 52 weeks and 1 day left

For that day to be sunday prob is $\frac{1}{7}$

$\therefore P(C/A) = \frac{1}{7}$

For $P(C/B)$ = leap year

= 366 days

$\frac{366}{7}$ = 52 weeks and 2 days left.

2 days can be {(sun, Mon), (Mon, Tue), (Tue, Wed), (Wed, thur), (thur, Fri), (Fri, Sat), (Sat, Sun)}

sunday has come two $P(C/B) = \frac{2}{7}$

$P(C) = P(A)\,P(C/A) + P(B)\,P(C/B)$

$= \frac{3}{4} \times \frac{1}{7} + \frac{1}{4} \times \frac{2}{7}$

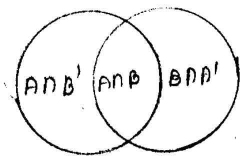
$= \frac{3}{28} + \frac{2}{28}$

$= \frac{5}{28}$

(2) (i) $P(A \cup B) = 0.9$

$P(A \cap B') = 0.4$

$P(B \cap A') = 0.3$



$P(A' \cup B') = P(A \cap B)'$, [De Morgan's law]

$= 1 - P(A \cap B) \longrightarrow (I)$

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$P(A\cap B') + P(A\cap B) + P(B\cap A') = P(A\cup B)$

$\therefore\ 0.4 + P(A\cap B) + 0.3 = 0.9$

$\therefore\ P(A\cap B) = 0.2$

Substituting in (I)

$P(A'\cup B') = 1 - 0.2$

$= 0.8$

(ii) $A\cup B \subset U$

$\therefore\ P(A\cup B) \le P(U)$

$\therefore\ P(A\cup B) \le 1$

$\therefore\ P(A) + P(B) - P(A\cap B) \le 1$

$\therefore\ P(A) + P(B) - 1 \le P(A\cap B)$

$\therefore\ P(A\cap B) \geq P(A) + P(B) - 1$

[D]

(1)

RAM	ROM
→ Random accesstheory	→ Read only memory
→ Information and program-me store in RAM is erased or destroyed if the current is cut off and it cannot be retired.	→ It is Internal memory of computer Information and programme stored in ROM is not erased or destroyed even when the current is cut off.

(2) $(216.444)_8 = (?)_2 = (?)_{10}$

$2 = (010)_2$

$1 = (001)_2$

$6 = (110)_2$

$4 = (100)_2$

$\therefore\ (216.444)_8 = (010001110.100100100)_2$

$= (10001110.1001001)_2$

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$$(216.444)_8 = 2\times 8^2 + 1\times 8^1 + 6\times 8^0 + 4\times 8^{-1} + 4\times 8^{-2} + 4\times 8^{-3}$$

$$= 128 + 8 + 6 + 4\times\frac{1}{8} + 4\times\frac{1}{64} + 4\times\frac{1}{512}$$

$$= 142 + 0.5 + 0.0625 + 0.0078125$$

$$= 142.5703125.$$

$$(216.444)_8 = (142.5703125)_{10}.$$

--- x --- x ---