Model Paper - 4

Std : XII.
Time : 3 hrs. **Maximum Marks : 75**

Instructions :

(1) There are **FIVE** questions in this question paper. Each question carries 15 marks and **all** are compulsory.
(2) Figures to the right indicate full marks.

Que : 1.

(A)

1. If a, b $\in$ R then we can find a δ_1-neighbourhood N(a, δ_1) of a and δ_2-neighbourhood N(b, δ_2) of b then prove that N(a, δ_1) $\cap$ N(b, δ_2) = ϕ. **(2)**
2. Prove that $\lim_{n\to\infty} r^n = 0, |r| < 1$. **(2)**

(B) Answer any Two. **(4)**

1. $\lim_{x\to\frac{\pi}{3}} \dfrac{\sin x - \sqrt{3}\cos x}{x - \frac{\pi}{3}}$.
2. $\lim_{x\to 0} \dfrac{15^x - 5^x - 3^x + 1}{x\log(1+x)}$.
3. $\lim_{x\to 2} \dfrac{\sqrt[3]{3x+2} - 2}{\sqrt[6]{x+62} - 2}$.

(C)

1. Find $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x - \sin x}}$. **(2)**
2. **Answer any Two.** **(2)**
   (i) Find $\lim_{x\to 0} \dfrac{e^x}{e^{e^x}}$.
   (ii) Find $\lim_{n\to\infty}\left(\dfrac{1^2}{n^3} + \dfrac{2^2}{n^3} + \dots\dots\dots + \dfrac{n^2}{n^3}\right)$.
   (iii) Find $\lim_{x\to 2} \dfrac{x^6 - 64}{x^{\frac{3}{2}} - 2^{\frac{3}{2}}}$.

(D)

1. If $(a+bx)\cdot e^{\frac{y}{x}} = x$ then prove that $x^3\dfrac{d^2y}{dx^2} = \left(x\dfrac{dy}{dx} - y\right)^2$. **(2)**
2. $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots\dots$ then prove that $\dfrac{dy}{dx} = y$. **(1)**

Que : 2.

(A)

1. If $f:(a,b)\to R$ is differentiable at x, $\forall x \in (a,b)$ then prove that it is continuous at x. **(2)**
2. State the rule for derivative of a composite function. **(1)**
3. State the formula of length of subtangent and subnormal at point(x,y) **(1)**

(B) Answer any Two. **(4)**

1. If $y=\cot^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x\neq 0$ find $\frac{dy}{dx}$.
2. If $y=x^{x^x}$ find $\frac{dy}{dx}$.
3. If $x=at^2$ and $y=2at$ then prove that $4ax^3\left(\frac{d^2y}{dx^2}\right)^2+2xy\left(\frac{dy}{dx}\right)^3-5a^2=0$.

(C) Attempt any Two. **(4)**

1. The height of a tower was measured from a point 200 m away from the tower. The angle of elevation was measured as 30° but the true angle of the elevation from that point was 30° 12'. What error must have been committed in the calculation of the height of the tower ?
2. Prove that $\lambda_1 \neq \lambda_2$, then the curves $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$ and $\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$ intersect each other orthogonally.
3. Prove that if $x>0$ then $\frac{\log(1+x)}{x}$ is a decreasing function.
4. Prove that x^x is minimum when $x=\frac{1}{e}$.

(D)

1. Find the approximate value of $\log_{10} 999$. $(\log_{10} e=0.4343)$. **(1)**
2. If there is 4% error in the measure of radius of a sphere. Find the percentage error in its surface area . **(1)**
3. Verify M. V. theorem for $f(x)=\cos^{-1}x$, $x\in[-1, 0]$. **(1)**

Que : 3.

(A)

1. Prove that $\int \operatorname{cosec} x\, dx=\log|\operatorname{cosec} x-\cot x|+c$

$$=\log\left|\tan\frac{x}{2}\right|+c.$$

Or **(2)**

Prove that $\int e^{ax}\cos bx\, dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c$.

2. If f is even and continuous on the [–a, a] then prove that $\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)dx$. **(2)**

(B) Evaluate any Two. **(4)**

1. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$.
2. $\int \frac{dx}{5+4\cos x+3\sin x}$.
3. $\int \frac{\log x}{(1+\log x)^2}dx$.

(C) Evaluate any Two. **(4)**

1. $\int_a^b \cos x\, dx$. (as the limit of a sum).
2. $\int_{-1}^{3} |2x-1|\, dx$.
3. Prove that $\int_{-1}^{1} \frac{dx}{\sqrt{1-2\alpha x+\alpha^2}} = 2$.

(D)

1. Evaluate $\int (\sqrt{\tan x}+\sqrt{\cot x})dx$. **(2)**
2. Prove that $\int e^x[f(x)+f'(x)]dx = e^x f(x)+c$. **(1)**

Que : 4.

(A) Solve any Two. **(4)**

1. Prove that the differential equation of family of circle having centres on Y-axis and touching X-axis is $(x^2-y^2)\frac{dy}{dx}=2xy$.
2. Solve : $\frac{dy}{dx}+x\tan(y-x)=1$.
3. Solve : $\frac{dy}{dx}+2y=e^{-x}$.
4. A body moves in a straight line according to $\frac{ds}{dt}=s+1$ where s is the distance from the initial point at time t. s is in metres and t is in minutes. Find the time required to travel a distance of 99 m.

(B) Attempt any Two. **(4)**

1. Prove that the area of the region enclosed by the circle $x^2+y^2=64$ and parabola $y^2=12x$ is $\frac{16}{3}(4\pi+\sqrt{3})$.
2. Line x = c divides the area of the region bounded by $y^2=4x$ and x = 8 in two regions having equal areas. Find c.
3. Find the volume of the solid generated when the portion of the ellipse in upper semiplane of X-axis is revolved about X-axis (a > b).

(C) Attempt any Two. **(4)**

1. Probability distribution of a discrete random variable X is given below :

X = x	0	1	2	3	4	5
p(x)	0.03	0.06	0.13	0.20	0.31	0.27

Find the value of $P(X \geq 1)$ and $P(0 < X < 5)$.

2. Balaji and Niranjan agree to play a game of tossing balanced die. If an integer less than 3 is obtained on the die Balaji accepts to pay Rs. 5 to Niranjan. If an integer 3 or more is obtained on the die how much amount Niranjan should pay to Balaji so that the game becomes fair.
3. The mean of standard deviation of a binomial random variable X are 8 and 2 respectively. Find the parameters of the probability distribution of X and obtain the value of $P(X = 0)$ and $P(1 \leq X \leq 3)$:

(D)

1. Evaluate $\int \frac{dx}{1+2\cos\alpha x + x^2}$ (α is constant). **(1)**
2. Define Homogeneous differential equation and obtain its general solution. **(2)**

Que : 5

(A)

1. Define (i) Mutually exclusive event. (ii) Axiomatic definition of probability.

   Or **(2)**

   State and prove addition law of probability for three events.
2. State and prove multiplicative law of probability. **(2)**

(B) Attempt any Two. **(4)**

1. Ten students are sitting in a row. What is the probability that two specified students are sitting adjacent to each other ?
2. For three events A, B and C, $P(A) = P(B) = P(C) = p$, $P(A \cap B) = P(B \cap C) = P(A \cap C) = p^2$ and $P(A \cap B \cap C) = p^3$ where $0<p<1$. Examine whether A, B and C are mutually independent and find $P(A' \cap B' \cap C')$.
3. There are 6 black and 3 red balls in a box. 5 balls are selected at random without replacement. Find the probability that (i) exactly two balls are red and (ii) atleast two balls are red.

(C)

1. A die is constructed in such a way that the probability that integer obtained on its face when tossed in proportional to the square of that integer. Find the probability that an integer on the face of a die is even. **(2)**
2. A leap year is taken at random, what is the probability of getting 53 Sundays in it. **(1)**
3. A, B and C are independent events if $P(A) = 2P(B) = 4P(C) = 0.4$ then find $P(A \cup B \cup C)$. **(1)**

(D)

1. What is kilobyte ? **(1)**
2. Convert $(25.1875)_{10}$ into octal form. **(2)**

①

Maths II (051E) Solution paper set : 4

Model paper - 4

Q1

A. (1) $a, b \in R$, $a \neq b$, $|a-b| > 0$

let $\delta = \frac{|a-b|}{2}$, $\delta > 0$

$\therefore 2\delta = |a-b|$

let $m_1 = N(a, \delta)$ and $m_2 = N(b, \delta)$

We shall prove that m_1 and m_2 are disjoint.

let $x \in m_1 \cap m_2 \Rightarrow x \in m_1$ and $x \in m_2$

$\Rightarrow |x-a| < \delta$ and $|x-b| < \delta$

now, $|a-b| = |a-x+x-b|$

$\leq |x-a| + |x-b|$

$< \delta + \delta$

$= 2\delta$

$\therefore |a-b| < 2\delta$

But $|a-b| = 2\delta$

$\therefore |a-b| < a-b$

Which is false Thus there can not be any element in $m_1 \cap m_2$ so m_1 & m_2 are disjoint.

$\therefore m_1 \cap m_2 = \phi$

let $\delta_1 \leq \delta$ and $\delta_2 \leq \delta$

$\therefore N(a, \delta_1) \subset N(a, \delta)$ and $N(b, \delta_2) \subset N(b, \delta)$

$\therefore N(a, \delta_1) \cap N(b, \delta_2) \subset N(a, \delta) \cap N(b, \delta)$

$\therefore N(a, \delta_1) \cap N(b, \delta_2) = \phi$ $\left(\because N(a, \delta) \cap N(b, \delta) = \phi\right)$

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(2) $\lim_{n \to \infty} r^n = 0$, $|r| < 1$

If $r = 0$ then $r^n = 0$ $\forall n$ Hence $\lim_{n \to \infty} r^n = 0$

suppose $r \neq 0$ and $|r| < 1$ then

$|r^n - 0| < \epsilon \Leftrightarrow |r^n| < \epsilon$

$\Leftrightarrow n \log |r| < \log \epsilon$

$\Leftrightarrow n > \frac{\log \epsilon}{\log |r|}$ $\left(\because |r| < 1 \Rightarrow \log |r| < 0\right)$

now if $\frac{\log \epsilon}{\log |r|} < 0$ then we can choose $m = 1$ because

$n \geq 1 \Rightarrow n > \frac{\log \epsilon}{\log |r|}$

so $|r^n - 0| < \epsilon$

and if $\frac{\log \epsilon}{\log |r|} > 0$ then we can take $m = \left[\frac{\log \epsilon}{\log |r|} + 1\right]$

so that if $m \in N$, $n \in N$, $n \geq m$

$$n > \frac{\log \epsilon}{\log |r|} \Rightarrow n \log |r| < \log \epsilon$$

$$\Rightarrow \log |r|^n < \log \epsilon$$

$$\Rightarrow |r|^n < \epsilon$$

$$\Rightarrow |r^n - 0| < \epsilon$$

In any case given any $\epsilon > 0$ we always find $m \in N$ such that $n \geq m$, $n \in N \Rightarrow |r^n - 0| < \epsilon$

$$\therefore \lim_{n \to \infty} r^n = 0$$

Q1.

(B) (1) $\lim_{x \to \frac{\pi}{3}} \frac{\sin x - \sqrt{3}\cos x}{x - \frac{\pi}{3}}$

$$= \lim_{x \to \frac{\pi}{3}} \frac{2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)}{x - \frac{\pi}{3}}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{2\left(\sin x \cos \frac{\pi}{3} - \cos x \sin \frac{\pi}{3}\right)}{x - \frac{\pi}{3}}$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{2 \sin\left(x - \frac{\pi}{3}\right)}{\left(x - \frac{\pi}{3}\right)}$$

$$= \lim_{\left(x - \frac{\pi}{3}\right) \to 0} 2 \frac{\sin\left(x - \frac{\pi}{3}\right)}{\left(x - \frac{\pi}{3}\right)}$$

$$= 2 \lim_{\left(x - \frac{\pi}{3}\right) \to 0} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\left(x - \frac{\pi}{3}\right)}$$

$$= 2 \times 1$$

$$= 2$$

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(2) $\lim_{x\to 0} \frac{15^x - 5^x - 3^x + 1}{x \log(1+x)}$

$= \lim_{x\to 0} \frac{5^x(3^x-1) - 1(3^x-1)}{x\log(1+x)}$

$= \frac{\lim_{x\to 0} \left(\frac{5^x-1}{x}\right)\left(\frac{3^x-1}{x}\right)}{\lim_{x\to 0} \frac{1}{x}\log(1+x)}$

$= \frac{\left(\lim_{x\to 0} \frac{5^x-1}{x}\right)\left(\lim_{x\to 0} \frac{3^x-1}{x}\right)}{\log\left(\lim_{x\to 0}(1+x)^{1/x}\right)}$

$= \frac{\log_e^5 \cdot \log_e^3}{\log_e^e} = \log_e^3 \cdot \log_e^5 .$

(3) $\lim_{x\to 2} \frac{\sqrt[3]{3x+2} - 2}{\sqrt[6]{x+62} - 2}$

$= \lim_{x\to 2} \frac{(3x+2)^{1/3} - 8^{1/3}}{(3x+2)-8} \cdot \frac{3[(x+62)-64]}{(x+62)^{1/6} - (64)^{1/6}}$

$= 3 \cdot \frac{\lim_{x\to 2} \frac{(3x+2)^{1/3} - 8^{1/3}}{(3x+2)-8}}{\lim_{x\to 2} \frac{(x+62)^{1/6} - (64)^{1/6}}{(x+62)-64}}$

$= 3 \frac{\lim_{(3x+2)\to 8} \frac{(3x+2)^{1/3} - 8^{1/3}}{(3x+2)-8}}{\lim_{(x+62)\to 64} \frac{(x+62)^{1/6} - (64)^{1/6}}{(x+62)-64}}$

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$$= \frac{3 \cdot \frac{1}{3} (8)^{1/3 - 1}}{\frac{1}{6} (64)^{1/6 - 1}}$$

$$= 3 \cdot \frac{6}{3} \frac{(2^3)^{-2/3}}{(2^6)^{-5/6}}$$

$$= 6 \frac{(2^{-2})}{(2^{-5})}$$

$$= 6 \cdot 2^3$$

$$= 48$$

(c)

(i) $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{\sin x}{x - \sin x}}$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{\sin x / x}{1 - \frac{\sin x}{x}}}$$

let $1 - \frac{\sin x}{x} = -t$

$\therefore 1 + t = \frac{\sin x}{x}$

As $x \to 0 \Rightarrow t \to 0$

$$= \lim_{t \to 0} (1+t)^{\frac{1+t}{-t}}$$

$$= \frac{1}{\lim_{t \to 0} (1+t)^{\frac{1}{t} + 1}}$$

$$= \frac{1}{\lim_{t \to 0} (1+t)^{1/t} \cdot \lim_{t \to 0} (1+t)}$$

$$= \frac{1}{e}$$

[5]

(2)

(i) $\lim_{x \to 0} \frac{e^x}{e^{e^x}}$

$= \frac{e^0}{e^{e^0}}$

$= \frac{1}{e^1} = \frac{1}{e}.$

(ii) $\lim_{n \to \infty} \frac{\Sigma n^2}{n^3}$

$= \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$

$= \lim_{n \to \infty} \frac{(\frac{1}{n}+1)(2+\frac{1}{n})}{6}$

$= \frac{(1+0)(2+0)}{6}$

$= \frac{1}{3}$

(iii) $\lim_{x \to 2} \frac{x^6 - 64}{x^{3/2} - 2^{3/2}}$

$= \frac{\lim_{x \to 2} \frac{x^6 - 2^6}{x-2}}{\lim_{x \to 2} \frac{x^{3/2} - 2^{3/2}}{x-2}}$ $(x \neq 2)$

$= \frac{6 \cdot (2)^5}{3/2 \cdot (2)^{3/2 - 1}} = \frac{12}{3} \cdot \frac{32}{\sqrt{2}}$

$= 2\sqrt{2} \cdot 32$

$= 64\sqrt{2}.$

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D (1) $(a+bx)e^{y/x} = x$

$\therefore e^{y/x} = \frac{x}{a+bx}$

$\therefore \frac{y}{x} = \log\left(\frac{x}{a+bx}\right)$

$\therefore \frac{y}{x} = \log x - \log(a+bx)$

diff w.r to x.

$\therefore \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}(b)$

$\therefore \frac{x\frac{dy}{dx} - y}{x^2} = \frac{a+bx-bx}{x(a+bx)}$

$\therefore x\frac{dy}{dx} - y = \frac{ax}{a+bx}$.

diff again w.r to x.

$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax b}{(a+bx)^2}$

$\therefore x\frac{d^2y}{dx^2} = \frac{a^2+abx-abx}{(a+bx)^2}$

$\therefore x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$

$\therefore x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2$

$\therefore x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

(7)

(2) $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

diff w.r to x.

$\therefore \frac{dy}{dx} = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots$

$= 1 + x + \frac{x^2}{2!} + \ldots$

$= y$

$\therefore \frac{dy}{dx} = y$.

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Q2)

(A)

(1) Now $f'(x)$ exists.

$\therefore \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ exists

Now $f(t) - f(x) = \left(\frac{f(t) - f(x)}{t - x}\right)(t - x)$ is a product of two functions both of which are defined in some deleted neighbourhood of x in (a, b) and limits of both exist as $t \to x$. Hence

$$\lim_{t \to x} [f(t) - f(x)] = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \lim_{t \to x} (t - x)$$

$$= f'(x) \cdot 0$$

$$= 0$$

Thus $\lim_{t \to x} f(t) = \lim_{t \to x} (f(t) - f(x) + f(x))$

$$= \lim_{t \to x} [f(t) - f(x)] + \lim_{t \to x} f(x)$$

$$= 0 + f(x)$$

$$= f(x)$$

$\therefore f$ is continuous at x

(2) Composite funtion.

If $f: (a, b) \to R$, $x \in (a, b)$ is differentiable at $x \in (a, b)$, if the range of f is a subset of the domain of another function g and $g: (c, d) \to R$ is differentiable at $f(x)$ then $g \circ f: (a, b) \to R$ is differentiable at x, and $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.

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3) length of subtangent $= \left| y / \frac{dy}{dx} \right|$

length of subnormal $= \left| y \cdot \frac{dy}{dx} \right|$

(B) Answer any Two

1) $y = \cot^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x \neq 0$.

Suppose $x = \tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \theta = \tan^{-1}x$.

$$y = \cot^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \cot^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \cot^{-1}\left(\frac{1-\cos\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}\right)$$

$$= \cot^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \cot^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)$$

$$= \cot^{-1}(\tan\theta/2)$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{2} - \theta/2\right)\right)$$

$-\frac{\pi}{2} < \theta < \pi/2$

$\Rightarrow -\frac{\pi}{4} < \theta/2 < \pi/4$

$\Rightarrow -\frac{\pi}{4} < -\theta/2 < \pi/4$

$\Rightarrow \frac{\pi}{4} < \frac{\pi}{2} - \theta/2 < \frac{3\pi}{4}$.

$\therefore y = \frac{\pi}{2} - \frac{\theta}{2}$

$\therefore y = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{-1}{2(1+x^2)}$

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(2) $y = x^{x^x}$

Taking log on both the sides

$\log y = x^x \log x$

$\therefore \log(\log y) = x\log x + \log(\log x)$

$\therefore \frac{1}{\log y} \cdot \frac{1}{y} \frac{dy}{dx} = \log x \cdot (1) + \frac{x}{x} + \frac{1}{\log x} \cdot \frac{1}{x}$

$\therefore \frac{dy}{dx} = y \log y \left(\log x + 1 + \frac{1}{x \log x}\right)$

$= x^{x^x} \cdot x^x \log x \left[\frac{x(\log x)^2 + x\log x + 1}{x \log x}\right]$

$\therefore \frac{dy}{dx} = x^{x^x} \cdot x^x \cdot \left(1 + x\log x + x(\log x)^2\right]$.

(3) $x = at^2$ & $y = 2at$

diff w.r to t

$\frac{dx}{dt} = 2at \neq 0$ $\quad \Big| \quad \frac{dy}{dt} = 2a$.

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $\left(\frac{dx}{dt} \neq 0\right)$

$= \frac{2a}{2at}$

$\therefore \frac{dy}{dx} = \frac{1}{t}$.

$\therefore \frac{d^2y}{dx^2} = \frac{-1}{t^2} \cdot \frac{dt}{dx}$

$= \frac{-1}{t^2} \cdot \frac{1}{2at}$

$= \frac{-1}{2at^3}$

(17)

$$L.H.S = 4ax^3\left(\frac{d^2y}{dx^2}\right)^2 + 2xy\left(\frac{dy}{dx}\right)^3 - 5a^2$$

$$= 4a(a^3t^6)\frac{1}{4a^2t^6} + 2(at^2)(2at)\cdot\frac{1}{t^3} - 5a^2$$

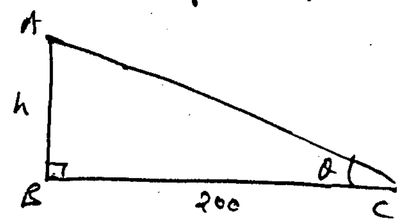
$$= a^2 + 4a^2 - 5a^2$$

$$= 0$$

$$= R.H.S.$$

(c)

(1) Suppose $\overline{AB}$ is a tower and C is the position on the ground from where the angle of elevation was measured



Here $\overline{AB} = h$ and $BC = 200$

and $\tan\theta = \frac{h}{200}$

$\therefore h = 200\tan\theta$

$\therefore \frac{dh}{d\theta} = 200\sec^2\theta$

also $\theta = 30° = \frac{\pi}{6}$ and $\delta\theta = 12' = \frac{12}{60}\cdot\frac{\pi}{180} = \frac{\pi}{900}$

$\therefore \frac{dh}{d\theta} = 200\left(\sec^2\frac{\pi}{6}\right) = 200 \times \left(\frac{2}{\sqrt{3}}\right)^2$

$= \frac{200 \times 4}{3} = \frac{800}{3}$

Now, the error in the calculation of the height of the tower is

$\delta h \approx \frac{dh}{d\theta}\cdot\delta\theta$

$= \frac{800}{3} \times \frac{\pi}{900}$

$= \frac{8\pi}{27}$ mts.

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(2) $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$ and $\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$

Let the point of intersection be (x,y) then solving the eqns we have

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}$$

$$\therefore \frac{x^2}{a^2+\lambda_1}-\frac{x^2}{a^2+\lambda_2}=\frac{y^2}{b^2+\lambda_2}-\frac{y^2}{b^2+\lambda_1}$$

$$\therefore x^2\left(\frac{a^2+\lambda_2-a^2-\lambda_1}{(a^2+\lambda_1)(a^2+\lambda_2)}\right)=\frac{y^2(b^2+\lambda_1-b^2-\lambda_2)}{(b^2+\lambda_2)(b^2+\lambda_1)}$$

$$\therefore \frac{x^2}{y^2}=\frac{(\lambda_1-\lambda_2)(a^2+\lambda_1)(a^2+\lambda_2)}{(\lambda_2-\lambda_1)(b^2+\lambda_2)(b^2+\lambda_1)}$$

$$\therefore \frac{x^2}{y^2}=\frac{-(a^2+\lambda_1)(a^2+\lambda_2)}{(b^2+\lambda_1)(b^2+\lambda_2)} \quad \text{—— (1)}$$

Now $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$

$$\therefore \frac{2x}{a^2+\lambda_1}+\frac{2y\frac{dy}{dx}}{b^2+\lambda_1}=0$$

$$\therefore \frac{2y\frac{dy}{dx}}{b^2+\lambda_1}=\frac{-2x}{a^2+\lambda_1}$$

$$\therefore m_1=\frac{dy}{dx}=\frac{(b^2+\lambda_1)x}{(a^2+\lambda_1)y}$$

$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$

$$\therefore \frac{2x}{a^2+\lambda_2}+\frac{2y\frac{dy}{dx}}{b^2+\lambda_2}=0$$

$$\therefore \frac{2y\frac{dy}{dx}}{b^2+\lambda_2}=\frac{-2x}{a^2+\lambda_2}$$

$$\therefore m_2=\frac{dy}{dx}=\frac{(b^2+\lambda_2)x}{(a^2+\lambda_2)y}$$

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Now, $m_1 \cdot m_2 = \dfrac{(b^2+\lambda_1)(b^2+\lambda_2)\, x^2}{(a^2+\lambda_1)(a^2+\lambda_2)\, y^2}$

$= \dfrac{(-y^2)\, x^2}{(x^2)\, y^2}$

$= -1.$

$\therefore$ The two curves intersect orthogonally.

(3) $f(x) = \dfrac{\log(1+x)}{x}, \quad x > 0$

$\therefore f'(x) = \dfrac{x \cdot \frac{1}{1+x} - \log(1+x) \cdot 1}{x^2}$

$= \dfrac{1}{x(1+x)} - \dfrac{\log(1+x)}{x^2}$ —— (1)

Suppose $g(x) = \log(1+x) - \dfrac{x}{1+x}$

$\therefore g'(x) = \dfrac{1}{1+x} - \dfrac{(1+x)\cdot 1 - x(1)}{(1+x)^2}$

$= \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}$

$= \dfrac{1+x-1}{(1+x)^2}$

$= \dfrac{x}{(1+x)^2} > 0$

$\therefore x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow \log(1+x) - \dfrac{x}{1+x} > 0$

$\Rightarrow \log(1+x) > \dfrac{x}{1+x}$

$\Rightarrow \dfrac{\log(1+x)}{x^2} > \dfrac{1}{x(1+x)}$

(14)

$\therefore f'(x) < 0$

$\therefore f(x) = \dfrac{\log(1+x)}{x}, x>0$ is a decreasing function

(4) Let $y = f(x) = x^x$

$\therefore \log y = x \log x$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \log x \cdot 1$

$\therefore f'(x) = \dfrac{dy}{dx} = x^x(1+\log x)$

Now $f'(x) = 0$ is necessary for f to have a max. & min values.

$\therefore x^x(1+\log x) = 0$

$\therefore 1+\log x = 0 \quad (x^x \neq 0)$

$\therefore \log_e^x = -1$

$\therefore x = e^{-1} = \dfrac{1}{e}$

Also $f''(x) = (1+\log x)\cdot\dfrac{d}{dx}(x^x) + x^x\cdot\dfrac{d}{dx}(1+\log x)$

$= (1+\log x)\cdot x^x\cdot(1+\log x) + x^x\left(\dfrac{1}{x}\right)$

$= x^x(1+\log x)^2 + x^{x-1}$

Now $f''\left(\dfrac{1}{e}\right) = \left(\dfrac{1}{e}\right)^{\frac{1}{e}}(1-1)^2 + \left(\dfrac{1}{e}\right)^{\frac{1}{e}-1}$

$= \left(\dfrac{1}{e}\right)^{(e^{-1}-1)} > 0$

$\therefore f\left(\dfrac{1}{e}\right)$ is minimum

Thus, x^x is minimum when $x = \dfrac{1}{e}$.

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(D)

(1) Let $f(x) = \log_{10}^{x}$

$= \frac{\log_e^x}{\log_e^{10}} = \log_e^x \cdot \log_{10}^e$

$\therefore f'(x) = \frac{\log_{10}^e}{x} = \frac{0.4343}{x}$

Now $f(1000) = \log_{10}^{1000}$

$= 3\log_{10}^{10} = 3$

$f'(1000) = \frac{0.4343}{1000} = 0.0004343$

$f(x+\delta x) \simeq f(x) + \delta x \cdot f'(x)$

$\therefore f(999) = f(1000-1)$

$\simeq f(1000) - 1 \cdot f'(1000)$

$= 3 - 0.0004343$

$= 2.9995657$

$\therefore \log_{10}^{999} \cong 2.9995657.$

(2) $\delta r = 4\%$ of r

$= \frac{4r}{100}$

Now surface area of Sphere is

$S = 4\pi r^2$ | $\frac{ds}{dr} = 8\pi r$

$\therefore \delta S \cong \frac{ds}{dr} \cdot \delta r$

$= 8\pi r \cdot \frac{4r}{100}$

$= \frac{32\pi r^2}{100} = \frac{8 \cdot 4\pi r^2}{100} = \frac{8}{100}$ of S.

$= 8\%$ of S. [16]

(3) . $f(x) = \cos^{-1}x$, $x \in [-1, 0]$.

Here, f is continuous on $[-1, 0]$ and diff. on $(-1, 0)$

$\therefore$ all the conditions of M.V. Theorem are satisfied

$\therefore$ we get some $c \in (-1, 0)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{\pi}{2} - \pi}{0 - (-1)}$$

$$\therefore \frac{-1}{\sqrt{1-c^2}} = -\frac{\pi}{2} \qquad \left(\because f'(x) = \frac{-1}{\sqrt{1-x^2}}\right)$$

$$\therefore \frac{1}{1-c^2} = \frac{\pi^2}{4}$$

$$\therefore 1 - c^2 = \frac{4}{\pi^2}$$

$$\therefore c^2 = 1 - \frac{4}{\pi^2}$$

$$\therefore c = \pm\sqrt{1 - \frac{4}{\pi^2}}$$

$\sqrt{1-\frac{4}{\pi^2}} \notin (-1, 0)$ but $-\sqrt{1-\frac{4}{\pi^2}} \in (-1, 0)$

$\therefore$ we get $c = -\sqrt{1-\frac{4}{\pi^2}}$ such that M.V. Theorem is satisfied.

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Q3

(A) (i) $\int \operatorname{cosec} x \, dx = \int \frac{dx}{\sin x}$

$= \int \frac{1+\tan^2 x/2}{2\tan^2 x/2} dx$

$= \int \frac{1}{2} \cot x/2 \, dx + \frac{1}{2}\int \tan\frac{x}{2} dx$

$= \frac{1}{2}\int \cot\frac{x}{2} dx + \frac{1}{2}\int \tan\frac{x}{2} dx.$

Both integrands are integrable on any interval for which $\frac{x}{2} \neq \frac{n\pi}{2}$, $x \neq n\pi$

$= \log|\sin x/2| - \log|\cos x/2| + C$

$= \log|\tan x/2| + c$

Also, $\log|\operatorname{cosec} x - \cot x| = \log\left|\frac{1-\cos x}{\sin x}\right|$

$= \log|\tan x/2|$

$\therefore \int \operatorname{cosec} x \, dx = \log|\tan\frac{x}{2}| + c$

$= \log|\sec x + \tan x| + c.$

OR

(i) $I = \int e^{ax} \cos bx \, dx$

$= \int \cos bx \cdot e^{ax} dx$

$= \cos bx \int e^{ax} dx - \int \left[\left(\frac{d}{dx}\cos bx\right) \int e^{ax} dx\right]$

(18)

$$= \cos bx \cdot \frac{e^{ax}}{a} + b\int \sin bx \cdot \frac{e^{ax}}{a} dx$$

$$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a}\int \sin bx \cdot e^{ax} dx$$

$$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a}\left[\sin bx \int e^{ax} dx - \int\left(\frac{d}{dx}\sin bx \int e^{ax} dx\right) dx\right]$$

$$= \cos bx \cdot \frac{e^{ax}}{a} + \frac{b}{a}\left[\sin bx \cdot \frac{e^{ax}}{a} - \int b\cos bx \cdot \frac{e^{ax}}{a} dx\right]$$

$$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a^2}\sin bx \cdot e^{ax} - \frac{b^2}{a^2}\int \cos bx \cdot e^{ax} dx$$

$$= \frac{e^{ax}}{a^2}\left[a\cos bx + b\sin bx\right] - \frac{b^2}{a^2} \cdot I + c'.$$

$$\therefore I\left(1+\frac{b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\cos bx + b\sin bx\right] + c'$$

$$\therefore I(a^2+b^2) = e^{ax}\left[a\cos bx + b\sin bx\right] + c'$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c$$

where $c = \frac{c'}{a^2+b^2}$

(2) since $-a < 0 < a$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx. \quad \text{——— (1)}$$

Let $x = -t$ in $I = \int_{-a}^{0} f(x)dx$.

$\therefore dx = -dt$

as $x \to -a \Rightarrow t \to a$

$x \to 0 \Rightarrow t \to 0$

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$$\therefore I = \int_{-a}^{0} f(x)\,dx$$

$$= \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)\,dt$$

$$= \int_{0}^{a} f(t)\,dt \quad (\because f \text{ is even function})$$

$$= \int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad (\because (1))$$

$$= 2\int_{0}^{a} f(x)\,dx.$$

(B)

(1) $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-b)\cdot\sin(x-a)}\,dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cdot\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\cdot\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \int \left(\cot(x-a) - \cot(x-b)\right) dx$$

$$= \frac{1}{\sin(a-b)} \left[\log|\sin(x-a)| - \log|\sin(x-b)|\right] + c$$

$$= \frac{1}{\sin(a-b)} \cdot \log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + c.$$

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(2) $\int \frac{dx}{5+4\cos x+3\sin x}$

let $\tan\frac{x}{2} = t$

$\therefore \sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\therefore (1+\tan^2\frac{x}{2})\,dx = 2dt$

$\therefore dx = \frac{2dt}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ & $\cos x = \frac{1-t^2}{1+t^2}$

$I = \int \frac{2dt/1+t^2}{5+4\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)}$

$= \int \frac{2dt}{5+5t^2+4-4t^2+6t}$

$= 2\int \frac{dt}{t^2+6t+9}$

$= 2\int \frac{dt}{(t+3)^2}$

$= 2\int (t+3)^{-2}\,dt$

$= 2\frac{(t+3)^{-1}}{-1} + c$

$= \frac{-2}{t+3} + c$

$= \frac{-2}{\tan x/2 + 3} + C$

[21]

(3) $\int \frac{\log x}{(1+\log x)^2} dx$

Let $\log_e x = t$

$\therefore x = e^t$

$dx = e^t dt$

$\therefore I = \int \frac{t \cdot e^t}{(1+t)^2} dt$

$= \int \left[\frac{(1+t)-1}{(1+t)^2}\right] e^t dt$

$= \int \left(\frac{1}{1+t} - \frac{1}{(1+t)^2}\right) e^t dt$

$= \int \left[\frac{1}{1+t} + \left(\frac{-1}{(1+t)^2}\right)\right] e^t dt$

$= \frac{e^t}{1+t} + c$

$= \frac{x}{1+\log x} + c$

(c)

(1) $\int_a^b \cos x \, dx$

$f(x) = \cos x$ is continuous in $[a, b]$. Divide $[a, b]$ in n subintervals of equal length. Each subinterval is of length. $h = \frac{b-a}{n}$. Therefore $nh = b-a$

$\therefore b = a + nh$

as $n \to \infty \Rightarrow h \to 0$

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$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} h \sum_{i=1}^{n} f(a+ih)$$

$$\therefore \int_a^b \cos x\,dx = \lim_{h\to 0} h \sum_{i=1}^{n} \cos(a+ih)$$

$$= \lim_{h\to 0} h\left[\cos(a+h) + \cos(a+2h) + \cos(a+3h) + \dots + \cos(a+nh)\right]$$

$$= \lim_{h\to 0} \frac{h}{2\sin h/2}\left[2\cos(a+h)\cdot\sin h/2 + 2\cos(a+2h)\cdot\sin\frac{h}{2} + 2\cos(a+3h)\cdot\sin\frac{h}{2} + \dots + 2\cos(a+nh)\cdot\sin\frac{h}{2}\right]$$

$$= \lim_{h\to 0}\frac{h/2}{\sin h/2}\cdot\lim_{h\to 0}\left[\sin\left(a+\frac{3h}{2}\right) - \sin\left(a+\frac{h}{2}\right) + \sin\left(a+\frac{5h}{2}\right) - \sin\left(a+\frac{3h}{2}\right) + \sin\left(a+\frac{7h}{2}\right) - \sin\left(a+\frac{5h}{2}\right) + \dots + \sin\left(a+nh+\frac{h}{2}\right) - \sin\left(a+nh-\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{h/2}{\sin h/2}\cdot\lim_{h\to 0}\left[\sin\left(b+\frac{h}{2}\right) - \sin\left(a+\frac{h}{2}\right)\right]$$

$$= 1.(\sin b - \sin a)$$

$$= \sin b - \sin a.$$

(2) $\int_{-1}^{3} |2x-1|\,dx$

$$[-1,3] = \left[-1,\frac{1}{2}\right] \cup \left[\frac{1}{2},3\right]$$

Now $|2x-1| = 2x-1$, if $x \geq \frac{1}{2}$ 25 [23]

$= -(2x-1)$, if $x < \frac{1}{2}$

$$\therefore I = \int_{-1}^{1/2} (1-2x)\,dx + \int_{1/2}^{3} (2x-1)\,dx$$

$$= \left[x - x^2\right]_{-1}^{1/2} + \left[x^2 - x\right]_{1/2}^{3}$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - (-1-1) + (9-3) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{17}{2}$$

(3) $$\int_{-1}^{1} \frac{dx}{\sqrt{1-2\alpha x+\alpha^2}}$$

$$= \frac{1}{-2\alpha} \int_{-1}^{1} (1-2\alpha x+\alpha^2)^{-1/2} (-2\alpha)\,dx$$

$$= \frac{1}{-2\alpha} \int_{-1}^{1} (1-2\alpha x+\alpha^2)^{-1/2} \cdot \frac{d}{dx}(1-2\alpha x+\alpha^2)\,dx$$

$$= -\frac{1}{2\alpha} \left[\frac{(1-2\alpha x+\alpha^2)^{1/2}}{1/2} \right]_{-1}^{1}$$

$$= -\frac{1}{2\alpha} \cdot 2\left[\sqrt{1-2\alpha x+\alpha^2} \right]_{-1}^{1}$$

$$= -\frac{1}{\alpha} \left[\sqrt{(1-\alpha)^2} - \sqrt{(1+\alpha)^2} \right]$$

$$= \frac{-1}{\alpha} \left[(1-\alpha) - (1+\alpha) \right]$$

$$= \frac{-1}{\alpha}(-2\alpha) = 2.$$

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(D)

(1) $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$

$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$

$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$

$\therefore I = \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

Let $\sin x - \cos x = t$

$\therefore (\cos x + \sin x) dx = dt$

$\therefore \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

$\therefore 1 - \sin 2x = t^2$

$\therefore 1 - t^2 = \sin 2x$

$\therefore I = \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}}$

$= \sqrt{2} \sin^{-1}(t) + c$

$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + c.$

(2) L.H.S. $= \int [f(x) + f'(x)] e^x dx$

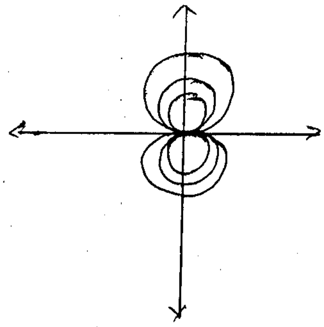
$= \int e^x f(x) dx + \int e^x f'(x) dx$

$= f(x) \int e^x dx - \int f'(x) \cdot e^x dx + \int e^x f'(x) dx + c$

$= f(x) \cdot e^x + c.$

Q4 (A)

(1) let the coordinate of the centre be (0, a) (a is the arbitary constant)



∴ the radius of the circle be |a|

∴ Equation of the circle is

$$(x-0)^2 + (y-a)^2 = a^2$$

$$\therefore x^2 + y^2 - 2ay + a^2 = a^2$$

$$\therefore x^2 + y^2 - 2ay = 0$$

$$\therefore \frac{x^2+y^2}{y} = 2a$$

Differentiate w.r.to x

$$\therefore \frac{y\frac{d}{dx}(x^2+y^2) - (x^2+y^2)\frac{dy}{dx}}{y^2} = 0$$

$$\therefore y\left(2x + 2y\frac{dy}{dx}\right) - x^2\frac{dy}{dx} - y^2\frac{dy}{dx} = 0$$

$$\therefore 2xy + 2y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - y^2\frac{dy}{dx} = 0$$

$$\therefore 2xy + y^2\frac{dy}{dx} - x^2\frac{dy}{dx} = 0$$

$$\therefore 2xy = (x^2-y^2)\frac{dy}{dx}$$

$$\therefore (x^2-y^2)\frac{dy}{dx} = 2xy$$

---

(2) $$\frac{dy}{dx} + x\tan(y-x) = 1$$

let $y - x = t$

$$\therefore \frac{dy}{dx} - 1 = \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{dt}{dx} + 1$$

$$\therefore \frac{dt}{dx} + 1 + x\tan t = 1$$

$$\therefore \frac{dt}{dx} + x\tan t = 0$$

$\therefore \frac{dt}{\tan t} + x\,dx = 0$

$\therefore \cot t\,dt + x\,dx = 0$

Integrate termwise

$\therefore \int \cot t\,dt + \int x\,dx = \log c$

$\therefore \log \sin t + \frac{x^2}{2} = \log c$

$\therefore \log \sin(y-x) + \log e^{\frac{x^2}{2}} = \log c$

$\therefore \log\left(\sin(y-x) \cdot e^{\frac{x^2}{2}}\right) = \log c$

$\therefore \sin(y-x)\, e^{\frac{x^2}{2}} = c$

This is the general solution and c is the arbitary constant.

---

(5) $\frac{dy}{dx} + 2y = e^{-x}$

This is the linear differential equation in the form of $\frac{dy}{dx} + Py = Q$

$\therefore P = 2$ and $Q = e^{-x}$

multiply both sides by $e^{\int P dx} = e^{\int 2 dx} = e^{2x}$

$\therefore e^{2x}\frac{dy}{dx} + 2y\,e^{2x} = e^{x}$

NOW $\frac{d}{dx}(y e^{2x}) = y e^{2x} \cdot 2 + e^{2x}\frac{dy}{dx}$

$\therefore \frac{d}{dx}(y e^{2x}) = e^{x}$

Integrate termwise

$y e^{2x} = \int e^{x}\,dx$

$\therefore y e^{2x} = e^{x} + c$

This is the general solution

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Q4 (A)

(4) $\frac{ds}{dt} = s+1$

$\therefore \frac{ds}{s+1} = dt$

Integrate termwise

$\int \frac{ds}{s+1} = \int dt$

$\therefore \log(s+1) = t + c$

When $t = 0$, $s = 0$

$\therefore c = \log 1$

$\therefore c = 0$

$\therefore \log(s+1) = t$

When $s = 99m$

$\therefore t = \log_e 100$

$= 2 \log_e 10$

$= 2(2.3026)$

$= 4.6052$

$\therefore t \cong 4.6$ minutes

Q4

B. Attempt any two

(1) Solving the equations

$x^2+y^2=64$ & $y^2=12x$

$\therefore x^2+12x-64=0$

$\therefore (x+16)(x-4)=0$

$\therefore x=-16$ or $x=4$

But $x=-16$ is not possible

$\therefore x=4$

$\therefore y^2=48$

$\therefore y = \pm\sqrt{48}$

$\therefore y = \pm 4\sqrt{3}$

$\therefore$ The intersecting points are $(4, 4\sqrt{3})$ and $(4, -4\sqrt{3})$

$x^2 + y^2 = 64$

$\therefore y = \sqrt{64 - x^2}$

$\alpha = \int_0^4 y\,dx$

$= \int_0^4 2\sqrt{3}\sqrt{x}\,dx$

$= 2\sqrt{3}\left[\frac{x^{3/2}}{3/2}\right]_0^4$

$= \frac{4\sqrt{3}}{3} \cdot 4^{3/2}$

$= \frac{32\sqrt{3}}{3}$

$y^2 = 12x$

$\therefore y = 2\sqrt{3}\sqrt{x}$

$\beta = \int_4^8 y\,dx$

$= \int_4^8 \sqrt{8^2 - x^2}\,dx$

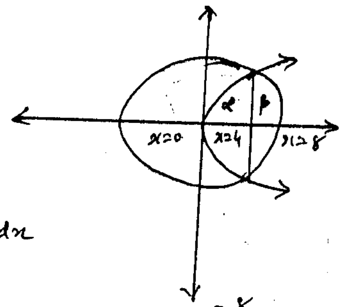
$= \left[\frac{x\sqrt{8^2 - x^2}}{2} + 32\sin^{-1}\left(\frac{x}{8}\right)\right]_4^8$

$= 32\sin^{-1}(1) - 2\sqrt{48} - 32\sin^{-1}\frac{1}{2}$

$= 32\frac{\pi}{2} - 2 \cdot 4\sqrt{3} - 32 \cdot \frac{\pi}{6}$

$= 32\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 8\sqrt{3}$

$= \frac{32\pi}{3} - 8\sqrt{3}$



Area of the required region

$= 2(\alpha + \beta)$

$= 2\left(\frac{32\sqrt{3}}{3} + \frac{32\pi}{3} - 8\sqrt{3}\right)$

$= \frac{16}{3}(\sqrt{3} + 4\pi)$

$\therefore$ Area $= \frac{16}{3}(4\pi + \sqrt{3})$

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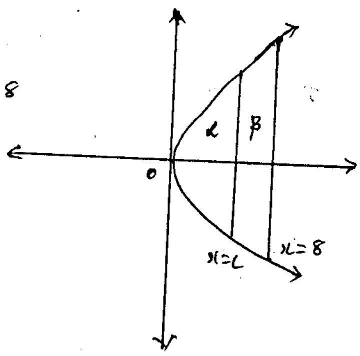
(2) The line $x = c$ divides the area of the region bounded by $y^2 = 4x$ and $x = 8$ into two equal region



$\therefore \alpha = \beta$

$\therefore \int_0^c y\,dx = \int_c^8 y\,dx$

$\therefore \int_0^c 2\sqrt{x}\,dx = \int_c^8 2\sqrt{x}\,dx$

$$\therefore \quad 2\left[\frac{x^{3/2}}{3/2}\right]_0^c = 2\cdot\left[\frac{x^{3/2}}{3/2}\right]_c^8$$

$$\therefore \quad \frac{2}{3}\cdot c^{3/2} = \frac{2}{3}\left[8^{3/2} - c^{3/2}\right]$$

$$\therefore 2c^{3/2} = 8^{3/2}$$

$$\therefore 2c^{3/2} = 2^{9/2}$$

$$\therefore c^{3/2} = 2^{7/2}$$

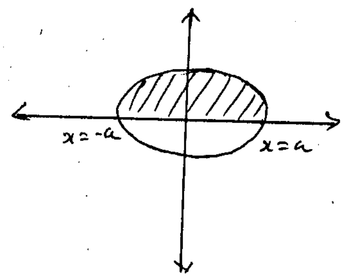
$$\therefore \boxed{c = 2^{7/3}}$$

---

3. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore y^2 = b^2\left(\frac{a^2 - x^2}{a^2}\right)$$

$$\therefore y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$



At x-axis, $y = 0$

$$a^2 - x^2 = 0$$

$$\therefore x^2 = a^2$$

$$\therefore x = \pm a$$

$$\text{volume} = \pi\int_a^b y^2\,dx$$

$$= \pi\int_{-a}^{a}\frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx$$

$$= \frac{2\pi b^2}{a^2}\left[a^2\int dx - \int x^2\,dx\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{2\pi b^2}{a^2}\cdot\frac{2a^3}{3}$$

$$= \frac{4\pi ab^2}{3}$$

Q4.

c.

(1) (i) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - 0.03$

$= 0.97$

$P(X \geq 1) = 0.97$

(ii) $P(0 < X < 5)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0.06 + 0.13 + 0.20 + 0.31$

$= 0.70$

$P(0 < X < 5) = 0.70$

(2) U = sample space of tossing a die

$\therefore U = \{1, 2, 3, 4, 5, 6\}$

$X : U \rightarrow R$

X(U) = Amount paid by Niranjan to Balaji

$X(U) = -5, \quad u = 1, 2$

$= a, \quad u = 3, 4, 5, 6$

$X = x$	-5	a
$P(x)$	$2/6$	$4/6$

$E(X) = \sum x_i P(x_i)$

$= \frac{-10}{6} + \frac{4a}{6}$

$= \frac{4a - 10}{6}$

The game of tossing a die is fair

$\therefore E(X) = 0$

$\therefore 4a - 10 = 0$

$\therefore 4a = 10$

$\therefore a = 5/2 \quad \therefore a = 2.5$

Niranjan should pay Rs 2.5 to Balaji

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Q4

c

3.

Mean $= np = 8$

varience $= npq = 4$

$\therefore \frac{npq}{np} = \frac{4}{8} = \frac{1}{2}$

$\therefore q = \frac{1}{2}$

$\therefore p = \frac{1}{2}$

$\therefore n = 16$

Binomial distribution of random variable x

$p(X=x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots n$

$p(X=x) = \binom{16}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{16-x}, \quad x = 0, 1, \ldots, 16$

$\therefore p(X=x) = \binom{16}{x} \left(\frac{1}{2}\right)^{16}$

(i) $p(X=0) = \binom{16}{0} \left(\frac{1}{2}\right)^{16} = \frac{1}{2^{16}}$

(ii) $p(1 \le x \le 3)$

$= p(x=1) + p(X=2) + p(x=3)$

$= \frac{1}{2^{16}} \binom{16}{1} + \frac{1}{2^{16}} \binom{16}{2} + \frac{1}{2^{16}} \binom{16}{3}$

$= \frac{1}{2^{16}} \left[\binom{16}{1} + \binom{16}{2} + \binom{16}{3} \right]$

$= \frac{1}{2^{16}} [16 + 120 + 560]$

$= \frac{696}{2^{16}}$

$\therefore p(1 \le x \le 3) = \frac{696}{2^{16}}$

Q4.

(D.)

(1) $\int \frac{dx}{1+2\cos\alpha x+x^2}$

$= \int \frac{dx}{x^2+2\cos\alpha\, x+\cos^2\alpha+\sin^2\alpha}$

$= \int \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha}$

$= \frac{1}{\sin\alpha}\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)+c$

$= \operatorname{cosec}\alpha \tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)+c$

---

(2)

let $M\,dx+N\,dy=0$ be the first order and first degree differential equation where M and N are homogeneous function of same degree. If this equation can be written in the form of $\frac{dy}{dx}=f(y/x)$. This is called homogeneous differential equation.

let $\frac{dy}{dx}=f(y/x)$ be the homogeneous differential equation

let $y/x=v$ $\therefore$ $y=vx$

differentiate w r to x

$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$v+x\frac{dv}{dx}=f(v)$

$\therefore x\frac{dv}{dx}=f(v)-v$

$\therefore \frac{dv}{f(v)-v}=\frac{dx}{x}$ (By variable seperable)

Integrate termwise

$\int\frac{dv}{f(v)-v}=\int\frac{dx}{x}+c$

$\therefore F(v)=\log|x|+c$

$\therefore F(y/x)=\log|x|+c$ where $\int\frac{dv}{f(v)-v}=F(v)$

This is the solution of homogeneous differential equation.

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Q5

(1) Define-

(i) Mutually exclusive event

Let A and B be events. If $A \cap B = \phi$ then A and B are called mutually exclusive events

(ii) Axiomatic defination of probability

Let U be a finite sample space and S be the power set of U. Let the set function $P: S \to R$ satisfy the following axioms:

Axiom 1 - For every $A \in S$, $P(A) \geq 0$

Axiom 2 - $P(U) = 1$

Axiom 3 - If $A_1 \in S$ and $A_2 \in S$ are any mutually exclusive events then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Set function P defined on S and satisfying the axioms 1, 2 and 3 is called a probability function and for $A \in S$, $P(A)$ is called probability of event A.

OR

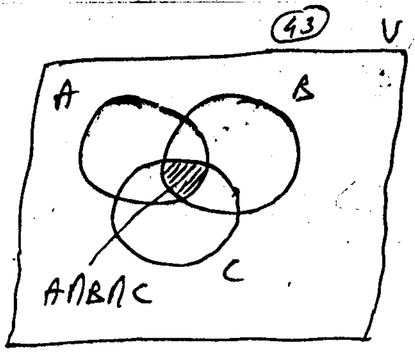
(1) Addition law of probability for three events

Statement - If A, B and C are events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

(43)

Proof – $P(A \cup B \cup C)$



$= P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$ ($\because$ distributive law)

$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

(2) <u>Multiplicative law of probability</u>

→ If A and B are events and $P(B) \neq 0$ then

$P(A \cap B) = P(B) \cdot P(A/B)$

Proof – Acc. to the defination of conditional probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{where } P(B) > 0$$

Now multiplying on both the sides by $P(B) > 0$.

$\therefore P(A \cap B) = P(B) \cdot P(A/B)$

→ If A_1, A_2, A_3 are three events then the probability $A_1 \cap A_2 \cap A_3$ of events A_1, A_2, A_3 to occur simultaneously is

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2).$$

Q5.

(B)

(1) 10 students are sitting in a row can be arranged in $^{10}P_{10}$ ways. $\therefore n(U) = {}^{10}P_{10} = 10!$

$\therefore n = 10!$

A = Event that two specified students among 10 are sitting adjacent to each other

$\therefore n(A) = 2 \cdot 9!$

$\therefore r = 2 \cdot 9!$

$p(A) = \frac{r}{n}$

$= \frac{2 \cdot 9!}{10!}$

$= \frac{2 \cdot 9!}{10 \cdot 9!}$

$= \frac{2}{10}$

$= \frac{1}{5}$

---

(2) $p(A) = p(B) = p(C) = p$

$p(A \cap B) = p(B \cap C) = p(C \cap A) = p^2$

$\therefore p(A \cap B \cap C) = p^3$ & $0 < p < 1$

Since $p(A \cap B) = p^2 = p \cdot p = p(A) \cdot p(B)$

$p(B \cap C) = p^2 = p \cdot p = p(B) \cdot p(C)$

$p(C \cap A) = p^2 = p \cdot p = p(C) \cdot p(A)$

$p(A \cap B \cap C) = p^3 = p \cdot p \cdot p = p(A) \cdot p(B) \cdot p(C)$

$\therefore$ A, B and C are mutually independent events.

(3) 6 Block and 3 Red balls.

5 balls are selected from 9 balls in $\binom{9}{5}$ ways $\therefore n = \binom{9}{5}$

A = Event that selecting exactly 2 red balls out of 5 selected balls.

$$\therefore P(A) = \frac{\binom{3}{2}\binom{6}{3}}{\binom{9}{5}}$$

$$= \frac{\binom{3}{1}\binom{6}{3}}{\binom{9}{4}}$$

$$= \frac{3 \times 20}{126}$$

$$= \frac{10}{21}$$

$$\therefore P(A) = \frac{10}{21}$$

B : Event that selecting atleast two out of five selected balls are red

$$= \frac{\binom{3}{2}\binom{6}{3} + \binom{3}{3}\binom{6}{2}}{\binom{9}{5}}$$

$$= \frac{60 + 15}{126}$$

$$= \frac{75}{126}$$

$$= \frac{25}{42}$$

Q5. (C) Ai : Event that getting integer on the face of the die is i

$P(A_i) \propto i^2$

$\therefore P(A_i) = k i^2$

$\therefore P(A_1) = k$ $\quad$ $P(A_4) = 16k$

$P(A_2) = 4k$ $\quad$ $P(A_5) = 25k$

$P(A_3) = 9k$ $\quad$ $P(A_6) = 36k$

$$\sum_{i=1}^{6} P(A_i) = 1$$

$\therefore P(A_1)+P(A_2)+P(A_3)+P(A_4)+P(A_5)+P(A_6) = 1$

$\therefore k + 4k + 9k + 16k + 25k + 36k = 1$

$\therefore 91k = 1$

$\therefore k = \frac{1}{91}$

$\therefore P(A_2) = \frac{4}{91} \quad P(A_4) = \frac{16}{91} \quad P(A_6) = \frac{36}{91}$

$P[A_2 \cup A_4 \cup A_6]$

$= P(A_2) + P(A_4) + P(A_6)$

$= \frac{4}{91} + \frac{16}{91} + \frac{36}{91}$

$= \frac{56}{91}$

$\therefore P$ (Event that getting even integer) $= \frac{56}{91}$

---

(2) In a leap Year 366 days

For 364 days there are 52 sundays

$\therefore$ the sample space of remaining two days

$U = \{$ (sun, mon), (mon, Tue), (Tue, wed), (wed, Th), (Th, FRI) (FRI, SAT), (SAT, SUN)$\}$

$\therefore n(U) = 7$

$\therefore n = 7$

A = Event that in a leap year there are 53 sundays

$\therefore A = \{$ (sun, mon), (sat, sun)$\}$

$\therefore n(A) = 2 \quad \therefore r = 2$

$P(A) = r/n$

$\therefore P(A) = \frac{2}{7}$

Q5

C.

(3) $P(A) = 2P(B) = 4P(C) = 0.4$

$\therefore P(A) = 0.4, \quad P(B) = 0.2, \quad P(C) = 0.1$

$$P(A \cup B \cup C) = 1 - P(A \cup B \cup C)'$$
$$= 1 - P(A' \cap B' \cap C')$$
$$= 1 - P(A')\,P(B')\,P(C')$$
$$= 1 - (0.6)(0.8)(0.9)$$
$$= 1 - 0.432$$
$$= 0.568$$

$P(A \cup B \cup C) = 0.568$

Q5

D.

(1) one kilobyte = 2^{10} = 1024 bytes

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(2) $(25.1875)_{10}$ convert into octal form

	Quotient	Remainder
25 ÷ 8	3	1 ↑
3 ÷ 8	0	3

$(25)_{10} = (31)_8$

Product of the fractional part by 8	Integral part
.1875 × 8 = 1.5000	1 ↓
.5000 × 8 = 4.0000	4

$(.1875)_{10} = (.14)_8$

$\therefore (25.1875)_{10} = (31.14)_8$