Model Paper – 5

Std : XII.
Time : 3 hrs. **Maximum Marks : 75**

Instructions :

(1) There are **FIVE** questions in this question paper. Each question carries 15 marks and **all** are compulsory.
(2) Figures to the right indicate full marks.

Que : 1.

(A)

1. Prove that $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\ n\in N$. **(2)**
2. Prove that $\lim_{h\to 0}\frac{a^h-1}{h}=\log_e a,\ (a>0)$. **(2)**

(B) Answer any Two. **(4)**

1. $\lim_{x\to\sqrt{3}}\frac{\tan^{-1}x-\frac{\pi}{3}}{x-\sqrt{3}}$. 2. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}9^{\frac{r}{n}}$. 3. $\lim_{x\to 0}\frac{x\sin x}{e^x-2+e^{-x}}$.

(C)

1. Find $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$. **(2)**
2. **Answer any Two.** **(2)**

(i) If $f(\theta) = (1+3\theta)^{\frac{1}{\theta}},\ \theta\neq 0$.
$= e^{\alpha},\ \theta=0$. is continuous $\theta = 0$, find α.

(ii) Find $\lim_{x\to 0}\frac{2^{5x}-2^{3x}}{\sin x}$.

(iii) Find $\lim_{n\to\infty}\frac{4^{n+1}-3^n}{4^n+3^{n+1}}$.

(D)

1. If ky = sin(x + y) then prove that $y_2 + (1 + y_1)^2 y=0$. **(2)**
2. Find $\frac{d}{dx}[\cos^2(x^2)-\sin^2(x^2)]$. **(1)**

Que : 2.

(A)

1. State and prove product rule of differentiation. **(2)**
2. State chain rule. **(1)**
3. State Rolle's theorem. **(1)**

(B) Evaluate any Two. **(4)**

1. $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$. 2. $\int \log\left(x + \sqrt{x^2 + a^2}\right) dx$. 3. $\int \frac{x^2}{(2x^2+1)(x^2-1)} dx$.

(C) Evaluate any Two. **(4)**

1. $\int_1^2 6^x dx$. (as the limit of a sum).

2. Find $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$.

3. Prove that $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$.

(D)

1. Evaluate : $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx$. **(2)**

2. Prove that $\int (f(x) + f'(x))\, e^x dx = f(x)e^x + C$. **(1)**

Que : 4.

(A) Answer any Two. **(4)**

1. Solve $e^{\frac{y}{x}}\left(x\frac{dy}{dx} - y\right) + x = 0$.
2. Solve $\frac{dy}{dx} = (9x + y + 1)^2$
3. Solve $\frac{dy}{dx} + y = e^x$ and if x = 0 and y = 1 then find the particular solution.
4. At any instant rate of decay of radium is proportional to its mass present. If the masse at time t_1 and t_2 are m_1 and m_2 then the time required to make the mass half of its original mass is $\frac{(t_2 - t_1)\log_e 2}{\log_e\left(\frac{m_1}{m_2}\right)}$.

(B) Attempt any Two. **(4)**

1. Find the area of the region bounded by $x = 3y^2 - 9$, Y-axis and the lines $y = 0$ and $y = 1$.
2. Prove that the area of the region bounded by $y = \cos x$, $x = 0$ and $x = \pi$ is 2.
3. Prove the volume of the solid generated by revolving the region bounded by $y = x^2 + 1$ and $y = 2x + 1$ about X-axis is $\frac{104\pi}{15}$.

(C) Attempt any Two. **(4)**

1. A box contains 5 distinct balls of which 2 are white and 3 are red. Two balls are drawn at random from it without replacement. A random variable X is defined on a sample space U associated with this experiment as follows :
   For u$\in$ U, X(u) = Number of white balls in u. Find probability distribution of random variable X.

2. The mean and variance of a binomial random variable with parameters n and p are 4 and 2 respectively. Find n and p. Find also $P(X \le 3)$ (2)

2. What is the probability that a girl is selected ... children

2. Three numbers are selected from the numbers 1 to 20. Find the probability that they are consecutive numbers. (1)

3. The probability function of a binomial distribution is $p(x)=\binom{6}{x}p^x q^{6-x}$, $x=0,1,2....6$. If $3p(2)=2p(3)$ find the value of p. (1)

3. For events A and B, if $P(A \cap B) = 0.1$ and $P(A' \cap B') = 0.4$ and $P(A' \cup B') = 0.6$ then find $P(A \cup B)$. (1)

(D)

1. Evaluate $\int (x^3 \; 7x \; 2x) \, dx$. (1)

1. State the names of the important languages which are used in computers. (1)

2. State the applications of differential equation. (1)

2. Convert $(1101.101)_2$ into decimal and octal number. (2)

3. Obtain the general solution of linear differential equation. (1)

Que : 5.

(A)

1. Define : (i) Exhaustive events. (ii) Elementary events.
   Or **(2)**
   For $A \subset B$ then prove that $P(B - A) = P(B) - P(A)$ and hence prove that $P(A) \le P(B)$.
2. If A and B are independent events then prove that A', B and A, B' are independent events. **(2)**

(B) Attempt any Two. **(4)**

1. In a lottery, tickets numbered from 1 to 4000 are sold. Find the probability that a prize winning ticket selected in a draw bears a number which is divisible by 10 or 25.
2. Suppose that 3 out of 100 men and 3 out of 1000 women in a city suffer from colour blindness. A random selected person of the city is found to be colour-blind. If the person is a man or women is assumed to be equally likely, what is the probability that the selected person is a woman?
3. A box contains r red and g green balls. Two balls are selected one by one at random without replacement. Find the probability that (i) the first ball is green and the second ball is red and (ii) the first ball is green and the second ball drawn is also green.

(C)

1. Three families having 4 children each in one family there is 1 boy and 3 girls in 2nd family there are 2 boys and 2 girls and in 3rd family there are 3 boys and 1 girl . One child is selected at random from each family then

Solution of paper set. No. V 1

Maths. II (051)(E)

Que. 1.

(A) (1) $\lim_{x \to a} \frac{x^n - a^n}{x-a}$

$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}.a + x^{n-3}.a^2 + \cdots + a^{n-1})}{(x-a)}$

$(a \notin D_f \therefore x-a \neq 0)$

$= \lim_{x \to a} x^{n-1} + a \cdot \lim_{x \to a} x^{n-2} + a^2 \cdot \lim_{x \to a} x^{n-3} + \cdots + a^{n-1}$

$= a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \cdots + a^{n-1}$

$= a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}$

$= n \cdot a^{n-1}$

(2) If $a = 1$, both the sides are zero.

let $a \neq 1$

let $a^h - 1 = x \quad \therefore a^h = x + 1$

We have assumed that the exponential function is continuous.

$\therefore \lim_{h \to 0} a^h = a^0 = 1.$

$\therefore \lim_{h \to 0} x = \lim_{h \to 0} a^h - 1 = 0.$

$\therefore$ As $h \to 0 \Rightarrow x \to 0$.

$\therefore \lim_{h \to 0} \frac{a^h - 1}{h} = \lim_{x \to 0} \frac{x}{\log_a(x+1)}$

$= \frac{1}{\lim_{x \to 0} \frac{1}{x} \log_a(1+x)}$

$= \frac{1}{\log_a \lim_{x \to 0} (1+x)^{1/x}}$ ($\because$ log is continuous)

$= \frac{1}{\log_a e}$

$= \log_e a$

2.

(B) Answer any two.

(1) $\lim_{x \to \sqrt{3}} \dfrac{\tan^{-1}x - \pi/3}{x - \sqrt{3}}$

$= \lim_{x \to \sqrt{3}} \dfrac{\tan^{-1}x - \tan^{-1}\sqrt{3}}{x - \sqrt{3}}$

$= \lim_{x \to \sqrt{3}} \dfrac{\tan^{-1}\left(\frac{x-\sqrt{3}}{1+\sqrt{3}x}\right)}{\frac{x-\sqrt{3}}{1+\sqrt{3}x} \times (1+\sqrt{3}x)}$

$= \lim_{x \to \sqrt{3}} \dfrac{\tan^{-1}\left(\frac{x-\sqrt{3}}{1+\sqrt{3}x}\right)}{\left(\frac{x-\sqrt{3}}{1+\sqrt{3}x}\right)} \times \dfrac{1}{\lim_{x \to \sqrt{3}} (1+\sqrt{3}x)}$

$= 1 \times \dfrac{1}{1+\sqrt{3}\cdot\sqrt{3}}$

$= \dfrac{1}{1+3}$

$= \dfrac{1}{4}.$

$\Big[\because \lim_{x \to \sqrt{3}} \dfrac{\tan^{-1} \frac{x-\sqrt{3}}{1+\sqrt{3}x}}{\frac{x-\sqrt{3}}{1+\sqrt{3}x}}$

let $\tan^{-1} \dfrac{x-\sqrt{3}}{1+\sqrt{3}x} = \alpha$

$\therefore \tan\alpha = \dfrac{x-\sqrt{3}}{1+\sqrt{3}x}$

As $x \to \sqrt{3} \Rightarrow \alpha \to 0$.

$= \lim_{\alpha \to 0} \dfrac{\alpha}{\tan\alpha}$

$= \dfrac{1}{\lim_{\alpha \to 0} \frac{\tan\alpha}{\alpha}}$

$= 1 \Big]$

(2) $\lim_{n \to \infty} \sum \frac{1}{n} \cdot 9^{r/n}$

$= \lim_{n \to \infty} \frac{1}{n} \left[9^{1/n} + 9^{2/n} + 9^{3/n} + \cdots + 9^{n/n} \right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{9^{1/n}\left((9^{1/n})^{n}-1\right)}{(9^{1/n}-1)}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[9^{1/n}\cdot\frac{8}{9^{1/n}-1}\right]$$

let $\frac{1}{n} = y$

As $n \to \infty \Rightarrow y \to 0$.

$$= \lim_{y\to 0} y \cdot \frac{9^{y}\cdot 8}{9^{y}-1}$$

$$= \frac{\lim\limits_{y\to 0} 9^{y}\cdot 8}{\lim\limits_{y\to 0} \frac{9^{y}-1}{y}}$$

$$= \frac{8\cdot 9^{0}}{\log_e 9}$$

$$= 8\cdot \log_9 e$$

(3). $$\lim_{x\to 0} \frac{x\cdot \sin x}{e^{x}-2+e^{-x}}$$

$$= \lim_{x\to 0} \frac{x\cdot \sin x}{e^{x}-2+\frac{1}{e^{x}}}$$

$$= \lim_{x\to 0} \frac{x\sin x\cdot e^{x}}{e^{2x}-2\cdot e^{x}+1}$$

$$= \lim_{x\to 0} \frac{x\sin x\cdot e^{x}}{(e^{x}-1)^{2}}$$

Dividing numerator and denominator by x^2

$$= \frac{\lim\limits_{x\to 0} \frac{\sin x}{x}\cdot e^{x}}{\left(\lim\limits_{x\to 0} \frac{e^{x}-1}{x}\right)^{2}}$$

$$= \frac{1\times e^{0}}{1}$$

$$= 1$$

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(C) (1) $\lim_{x \to \pi/4} \dfrac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2}$

$= \lim_{x \to \pi/4} \dfrac{\sqrt{2} - \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)}{16\left(x - \pi/4\right)^2}$

$= \lim_{x \to \pi/4} \dfrac{\sqrt{2} - \sqrt{2}\left(\cos x \cdot \cos \pi/4 + \sin x \cdot \sin \pi/4\right)}{16\left(x - \pi/4\right)^2}$

$= \lim_{x \to \pi/4} \dfrac{\sqrt{2} - \sqrt{2}\cos\left(x - \pi/4\right)}{16\left(x - \pi/4\right)^2}$

let $x - \pi/4 = t$

$\therefore$ As $x \to \pi/4 \Rightarrow t \to 0$.

$= \lim_{t \to 0} \dfrac{\sqrt{2} - \sqrt{2}\cos t}{16 t^2}$

$= \lim_{t \to 0} \dfrac{\sqrt{2}(1 - \cos t)(1 + \cos t)}{16(1 + \cos t) \cdot t^2}$

$= \dfrac{\sqrt{2}}{16} \lim_{t \to 0} \dfrac{\sin^2 t}{t^2} \times \dfrac{1}{\lim_{t \to 0}(1 + \cos t)}$

$= \dfrac{\sqrt{2}}{16} \left(\lim_{t \to 0} \dfrac{\sin t}{t}\right)^2 \times \dfrac{1}{\lim_{t \to 0}(1 + \cos t)}$

$= \dfrac{\sqrt{2}}{16} \times 1 \times \dfrac{1}{2}$

$= \dfrac{1}{16\sqrt{2}}$

(2) <u>Answer any two.</u>

(i) f is continuous at $\theta = 0$.

$\therefore \lim_{\theta \to 0} f(\theta) = f(0)$

$\therefore \lim_{\theta \to 0} (1 + 3\theta)^{1/\theta} = e^{\alpha}$

$\therefore \left(\lim_{\theta \to 0} (1 + 3\theta)^{1/3\theta}\right)^3 = e^{\alpha}$

$\therefore \quad e^3 = e^{\alpha}$

$\therefore \quad \boxed{\alpha = 3}$

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(ii) $\lim_{x \to 0} \frac{2^{5x} - 2^{3x}}{\sin x}$

$= \lim_{x \to 0} \frac{32^x - 1 + 1 - 8^x}{\sin x}$

$= \frac{\lim_{x \to 0} \frac{32^x - 1}{x} - \lim_{x \to 0} \frac{8^x - 1}{x}}{\lim_{x \to 0} \frac{\sin x}{x}}$

$= \frac{\log_e 32 - \log_e 8}{1}$

$= \log_e 32/8$

$= \log_e 4$

(iii) $\lim_{n \to \infty} \frac{4^{n+1} - 3^n}{4^n + 3^{n+1}}$

$= \lim_{n \to \infty} \frac{4^n . 4 - 3^n}{4^n + 3^n . 3}$

Dividing numerator and denominator by 4^n

$= \lim_{n \to \infty} \frac{4 - \left(\frac{3}{4}\right)^n}{1 + 3 \cdot \left(\frac{3}{4}\right)^n}$

$= \frac{4 - \lim_{n \to \infty} \left(\frac{3}{4}\right)^n}{1 + 3 \cdot \lim_{n \to \infty} \left(\frac{3}{4}\right)^n}$

$= \frac{4 - 0}{1 + 3 \cdot 0}$

$= 4$

(D) (1) $Ky = \sin(x + y)$

Differentiating w.r.t. x.

$\therefore \; Ky_1 = \cos(x + y) \cdot (1 + y_1)$

$\therefore \; \cos(x + y) = \frac{Ky_1}{1 + y_1}$

Again differentiating w.r.t. x.

6.

$$-\sin(x+y)(1+y_1) = k\left[\frac{(1+y_1)y_2 - y_1 \cdot y_2}{(1+y_1)^2}\right]$$

$$\therefore \; -ky(1+y_1)^2 = k\left[y_2 + \cancel{y_1 y_2} - \cancel{y_1 y_2}\right]$$

$$\therefore \; -\cancel{k}y(1+y_1)^2 = \cancel{k}\,y_2$$

$$\therefore \; y_2 + y(1+y_1)^2 = 0.$$

(2) $\frac{d}{dx}\left(\cos^2(x^2) - \sin^2(x^2)\right)$

$$= \frac{d}{dx}\left(\cos 2x^2\right)$$

$$= -\sin 2x^2 \cdot 4x$$

$$= -4x \cdot \sin 2x^2$$

$$\therefore \; \frac{d}{dx}\left(\cos^2(x^2) - \sin^2(x^2)\right) = -4x \sin 2x^2.$$

Q.2

(A) (1) Product rule of differentiation.

Statement : If f and $g : (a, b) \to R$ are both differentiable at $x \in (a, b)$ then $f \cdot g$ is also differentiable at x and

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x).$$

Proof: Suppose $m(x) = f(x)\cdot g(x)$.

Now, $\frac{d}{dx}\left(f(x)\cdot g(x)\right)$

$$= \frac{d}{dx}\left(m(x)\right)$$

$$= \lim_{t \to x} \frac{m(t) - m(x)}{t - x}$$

$$= \lim_{t \to x} \frac{f(t)\cdot g(t) - f(x)\cdot g(x)}{t - x}$$

$$= \lim_{t \to x} \frac{f(t)\cdot g(t) - f(t)\cdot g(x) + f(t)\cdot g(x) - f(x)\cdot g(x)}{t - x}$$

$$= \lim_{t \to x} f(t) \cdot \lim_{t \to x} \frac{g(t) - g(x)}{t - x} + g(x) \cdot \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x)).$$

[$\because$ f is differentiable, so f is continuous.
$\therefore \lim_{t \to x} f(t) = f(x)$]

Thus, $f \cdot g$ is differentiable at x.

$$\therefore \frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x)).$$

(2) Chain Rule:

If y is a differentiable function of z, and z is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

This rule is called the chain rule.

(3) Rolle's theorem:

If a real function f is continuous on $[a, b]$, differentiable on (a, b) and if $f(a) = f(b)$ then there must be some $x \in (a, b)$ such that $f'(x) = 0$.

(B) Answer any two:

(1) $f(x) = e^x + 1 \quad , \quad x \geq 0$

$\qquad\;\; = 2 - 3x \quad , \quad x < 0$.

$\lim_{x \to 0+} f(x)$	$\lim_{x \to 0-} f(x)$
$= \lim_{x \to 0+} e^x + 1$	$= \lim_{x \to 0-} 2 - 3x$
$= e^0 + 1$	$= 2 - 0$
$= 1 + 1$	$= 2$.
$= 2$	

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also, $f(0) = e^0 + 1$

$= 2$.

$\therefore \lim_{x \to 0_+} f(x) = \lim_{x \to 0_-} f(x) = f(0) = 2$.

$\therefore f(x)$ is continuous at $x = 0$.

Now, $f'(0) = \lim_{t \to 0} \dfrac{f(t) - f(0)}{t - 0}$.

$$\lim_{t \to 0_+} \frac{f(t) - f(0)}{t - 0}$$

$$= \lim_{t \to 0_+} \frac{e^t + 1 - 2}{t}$$

$$= \lim_{t \to 0_+} \frac{e^t - 1}{t}$$

$$= 1$$

$$\lim_{t \to 0_-} \frac{f(t) - f(0)}{t - 0}$$

$$= \lim_{t \to 0_-} \frac{2 - 3t - 2}{t}$$

$$= \lim_{t \to 0_-} -\frac{3t}{t}$$

$$= -3$$

$\therefore \lim_{t \to 0_+} \dfrac{f(t) - f(0)}{t - 0} \neq \lim_{t \to 0_-} \dfrac{f(t) - f(0)}{t - 0}$

$\therefore f'(0)$ does not exist.

$\therefore f(x)$ is not differentiable at $x = 0$.

(2). $x = a\left(\cos\theta + \log\left(\tan \theta/2\right)\right)$

Differentiating w.r.t. θ

$$\frac{dx}{d\theta} = a\left(-\sin\theta + \frac{1}{\tan \theta/2} \cdot \sec^2 \theta/2 \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{\cos\theta/2}{\sin\theta/2} \cdot \frac{1}{\cos^2\theta/2} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right)$$

$$= a\left(\frac{1 - \sin^2\theta}{\sin\theta}\right)$$

$$= \frac{a \cdot \cos^2\theta}{\sin\theta} \neq 0.$$

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$y = a \sin\theta$

Differentiating w.r.t. θ

$\therefore \frac{dy}{d\theta} = a\cos\theta$

$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, $\left(\frac{dx}{d\theta} \neq 0\right)$

$= \frac{a\cos\theta}{a\cos^2\theta/\sin\theta}$

$= \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{dy}{dx} = \tan\theta$

(3) $y = x^{x^{x \cdots \infty}}$

$\therefore y = x^{y}$ Where $y = x^{x^{x \cdots \infty}}$

Taking log on both sides,

$\log y = y \log x$

Differentiating w.r.t. x.

$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$.

$\therefore \frac{1}{y} \cdot \frac{dy}{dx} - \log x \cdot \frac{dy}{dx} = \frac{y}{x}$

$\therefore \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$

$\therefore \frac{dy}{dx}\left(\frac{1 - y\log x}{y}\right) = \frac{y}{x}$

$\therefore \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$

(c) Attempt any two:

$\Delta = \frac{1}{2} bc \sin A$

When $A = \pi/6$

$\Delta = \frac{1}{2} bc \cdot \frac{1}{2}$

$\therefore \Delta = \frac{bc}{4}$ ---------- (1)

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$$\Delta = \frac{1}{2} bc \sin A$$

$$\therefore \frac{d\Delta}{dA} = \frac{1}{2} bc \cdot \cos A$$

When $A = \pi/6$

$$\frac{d\Delta}{dA} = \frac{1}{2} bc \cdot \cos \pi/6$$

$$= \frac{1}{2} \cdot bc \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4} \cdot bc.$$

also, $\delta A = \frac{x}{100} \cdot \frac{\pi}{6}$

$$\therefore \delta\Delta \simeq \frac{d\Delta}{dA} \cdot \delta A$$

$$= \frac{\sqrt{3}}{4} \cdot bc \cdot \frac{x}{100} \cdot \frac{\pi}{6}$$

$$= \frac{\sqrt{3}\pi x}{6} \cdot \frac{1}{100} \cdot \frac{bc}{4}$$

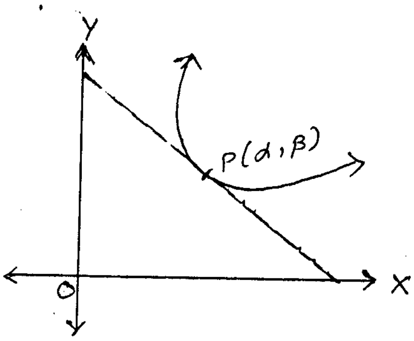
$$= \frac{\sqrt{3}\pi x}{6} \cdot \frac{1}{100} \cdot \Delta \qquad (\because \text{From } (1))$$

$\therefore \delta\Delta \cong \frac{\sqrt{3}\pi x}{6}$ % of its area.

(2) let (α, β) lies on this curve $x^{2/3} + y^{2/3} = a^{2/3}$

$$\therefore \alpha^{2/3} + \beta^{2/3} = a^{2/3} \text{ ---------- } (1)$$

Here, $x^{2/3} + y^{2/3} = a^{2/3}$



Differentiating w.r.t. x.

$$\frac{2}{3} x^{2/3 - 1} + \frac{2}{3} \cdot y^{2/3 - 1} \cdot \frac{dy}{dx} = 0.$$

$$\therefore \frac{2}{3} x^{-1/3} + \frac{2}{3} \cdot y^{-1/3} \cdot \frac{dy}{dx} = 0.$$

$$\therefore \frac{\cancel{2}}{\cancel{3}} x^{-1/3} = -\frac{\cancel{2}}{\cancel{3}} y^{-1/3} \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\therefore \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

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$\therefore$ Slope of tangent at $(\alpha, \beta) = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)}$

$$= -\frac{\beta^{1/3}}{\alpha^{1/3}}.$$

$\therefore$ Equation of tangent at (α, β) is

$$y - \beta = -\frac{\beta^{1/3}}{\alpha^{1/3}}(x - \alpha)$$

$$\therefore \alpha^{1/3}\cdot y - \beta\cdot\alpha^{1/3} = -\beta^{1/3}\cdot x + \alpha\cdot\beta^{1/3}$$

$$\therefore x\cdot\beta^{1/3} + \alpha^{1/3}\cdot y = \alpha\beta^{1/3} + \beta\alpha^{1/3}$$

$$\therefore x\beta^{1/3} + \alpha^{1/3}\cdot y = \alpha^{1/3}\cdot\beta^{1/3}\left(\alpha^{2/3} + \beta^{2/3}\right)$$

$$\therefore x\beta^{1/3} + \alpha^{1/3}\cdot y = \alpha^{1/3}\cdot\beta^{1/3}\cdot a^{2/3} \qquad (\because \text{From } ①)$$

The tangent meets the X-axis at A and Y-axis at B.

$\therefore$ At A, $y = 0$.

$$\therefore x = \alpha^{1/3}\cdot a^{2/3}$$

$$\therefore A\left(\alpha^{1/3}\cdot a^{2/3}, 0\right)$$

and at B, $x = 0$.

$$\therefore y = \beta^{1/3}\cdot a^{2/3}$$

$$\therefore B\left(0, \beta^{1/3}\cdot a^{2/3}\right)$$

$$\therefore AB = \sqrt{\alpha^{2/3}\cdot a^{4/3} + \beta^{2/3}\cdot a^{4/3}}$$

$$= \sqrt{a^{4/3}\left(\alpha^{2/3} + \beta^{2/3}\right)}$$

$$= \sqrt{a^{4/3}\cdot a^{2/3}}$$

$$= \sqrt{a^2}$$

$$= |a| = \text{constant}.$$

12.

(3) To prove $\frac{x}{1+x} < \log(1+x) < x$

let $f(x) = \log(1+x) - \frac{x}{1+x}$.

$\therefore f'(x) = \frac{1}{1+x} - \left[\frac{(1+x)-x}{(1+x)^2}\right]$

$= \frac{1}{1+x} - \frac{1}{(1+x)^2}$

$= \frac{1+x-1}{(1+x)^2}$

$= \frac{x}{(1+x)^2} > 0 \quad (\because x > 0)$

$\therefore f$ is increasing function for $x > 0$.

$\therefore x > 0$.

$\therefore f(x) > f(0)$

$\therefore \log(1+x) - \frac{x}{1+x} > 0$

$\therefore \log(1+x) > \frac{x}{1+x}$

$\therefore \frac{x}{1+x} < \log(1+x)$ ——— (1)

and, let $g(x) = x - \log(1+x)$

$\therefore g'(x) = 1 - \frac{1}{1+x}$

$= \frac{1+x-1}{1+x}$

$= \frac{x}{1+x} > 0 \quad (\because x > 0)$.

$\therefore g$ is increasing function for $x > 0$.

$\therefore x > 0$

$\therefore g(x) > g(0)$

$\therefore x - \log(1+x) > 0$

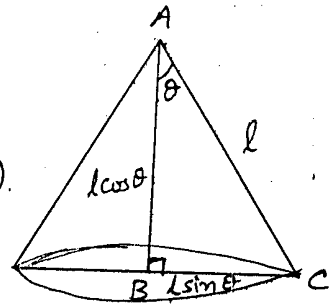
$\therefore x > \log(1+x)$

$\therefore \log(1+x) < x$ ——— (2)

From ① and ②, we get

$$\frac{x}{1+x} < \log(1+x) < x$$

(4) let the semi-vertical angle be θ, and Oblique height be l. (Constant).



$\therefore AB = l\cos\theta$,

$BC = l\sin\theta$

$\therefore$ Radius of the base $= l\sin\theta$

height $= l\cos\theta$

Let the volume be $f(\theta)$

$\therefore f(\theta) = \frac{1}{3}\pi r^2 h$

$\therefore f(\theta) = \frac{1}{3}\pi l^2 \sin^2\theta \cdot l\cos\theta$

$\therefore f(\theta) = \frac{1}{3}\pi l^3 \sin^2\theta \cdot \cos\theta$

$\therefore f'(\theta) = \frac{1}{3}\pi l^3 \left(\sin^2\theta(-\sin\theta) + \cos\theta \cdot 2\sin\theta\cos\theta\right)$

$= \frac{1}{3}\pi l^3 \sin\theta\left(2\cos^2\theta - \sin^2\theta\right)$

f to have maximum or minimum, $f'(\theta) = 0$.

$\therefore \frac{1}{3}\pi l^3 \sin\theta\left(2\cos^2\theta - \sin^2\theta\right) = 0$.

$\therefore 2\cos^2\theta = \sin^2\theta \quad (\because \sin\theta \neq 0)$

$\therefore \tan^2\theta = 2$

$\therefore \tan\theta = \sqrt{2}$

$\therefore \theta = \tan^{-1}\sqrt{2}$

$\tan\theta = \sqrt{2}$

$\therefore \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}, \quad \cos\theta = \frac{1}{\sqrt{3}}$

Again, $f'(\theta) = \frac{1}{3}\pi l^3 \sin\theta\left(2\cos^2\theta - \sin^2\theta\right)$

$\therefore f''(\theta) = \frac{1}{3}\pi l^3 \Big[\sin\theta\left(2\cdot 2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta\right)$

$+ \left(2\cos^2\theta - \sin^2\theta\right)\cdot\cos\theta\Big]$

$= \frac{1}{3}\pi l^3 \Big[-4\cos\theta\cdot\sin^2\theta - 2\sin^2\theta\cos\theta$

$+ 2\cos^3\theta - \sin^2\theta\cos\theta\Big]$

$$= \frac{1}{3}\pi l^3\left[2\cos^3\theta - 7\sin^2\theta\cdot\cos\theta\right]$$

$$= \frac{1}{3}\pi l^3\left[2\cdot\frac{1}{3\sqrt{3}} - 7\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{3}\pi l^3\left[\frac{-12}{3\sqrt{3}}\right] = -\frac{4}{3\sqrt{3}}\pi l^3 < 0$$

$\therefore$ Volume of this cone is maximum when the semi-vertical angle is $\tan^{-1}\sqrt{2}$

(D) (1) $T = 2\pi\sqrt{\dfrac{l}{g}}$

$$\log T = \log 2\pi + \frac{1}{2}\left[\log l - \log g\right]$$

$$\therefore \frac{1}{T}\cdot\frac{dT}{dl} = 0 + \frac{1}{2}\left[\frac{1}{l}\right]$$

$$\therefore \frac{dT}{dl} = \frac{T}{2l}$$

and $\delta l = \dfrac{4l}{100}$.

$$\therefore \delta T \cong \frac{dT}{dl}\cdot\delta l$$

$$= \frac{T}{2l}\times\frac{4l}{100}$$

$$= \frac{2}{100}\times T$$

$= 2\%$ of its T.

$\therefore$ Increase in the period is 2%.

(2) $y = e^{x/c}$

$$\therefore \frac{dy}{dx} = e^{x/c}\cdot\frac{1}{c}$$

$$\therefore \left(\frac{dy}{dx}\right)_{at\,(x,y)} = e^{x/c}\cdot\frac{1}{c}.$$

$\therefore$ length of subtangent $= \left|\dfrac{y}{dy/dx}\right|$

$$= \left|\frac{y}{e^{x/c}\cdot\frac{1}{c}}\right|$$

$$= |c| \quad (\because y = e^{x/c})$$

$=$ Constant

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(3) $f(x) = \cot x$

$\therefore f'(x) = -\operatorname{cosec}^2 x < 0 \quad \forall x \in R - \{k\pi / k \in z\}$.

$\therefore f$ is decreasing function on any interval not containing $k\pi$, $k \in z$.

$\therefore g(x) = \cot^{-1} x$

$\therefore g'(x) = -\frac{1}{1+x^2} < 0, \quad x \in R$

$\therefore g$ is decreasing function $\forall x \in R$.

Q.3 (A)

(1) $\int \sec x \, dx$.

$\sec x + \tan x \neq 0 \Rightarrow 1 + \sin x \neq 0$

$\Rightarrow x \neq (4n-1)\pi/2, n \in z$.

$$\therefore \int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

Now, $t = \sec x + \tan x$ is continuous and differentiable on any interval not containing $(2n-1)\pi/2$,

$$\therefore \frac{dt}{dx} = \sec^2 x + \sec x \cdot \tan x$$

$$= \sec x(\sec x + \tan x)$$

$\frac{dx}{dt} \neq 0 \Leftrightarrow x \neq (2n-1)\pi/2$

Thus, on any interval not containing $(2n-1)\pi/2$,

$\therefore \int \sec x \, dx = \log|\sec x + \tan x| + c$

$$\log|\sec x + \tan x| = \log\left|\frac{1+\tan^2 x/2}{1-\tan^2 x/2} + \frac{2\tan x/2}{1-\tan^2 x/2}\right|.$$

$$= \log\left|\frac{(1+\tan x/2)^2}{(1-\tan x/2)(1+\tan x/2)}\right|$$

$$= \log\left|\frac{1+\tan x/2}{1-\tan x/2}\right|$$

$$= \log\left|\tan\left(\pi/4 + x/2\right)\right|$$

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$\therefore \int \sec x\, dx = \log |\sec x + \tan x| + c$

$= \log |\tan(\pi/4 + x/2)| + c.$

OR

(1) Integration by parts:

Statement: If function f and g are differentiable on interval $I \subset R$, f' and g' are continuous on I, then

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx, \quad x \in I.$$

Proof: Here, f and g are differentiable function of x.

$\therefore$ according to working rule of differentiation,

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x) \text{ --------- } (1)$$

Also, according to given conditions,

f, g, f', g' are continuous on interval I.

$\therefore$ fg' and $f'g$ are also continuous and so they are integrable.

$\therefore$ According to the definition of integration from result (1),

$$f(x)\cdot g(x) = \int (f(x)\cdot g'(x) + f'(x)\cdot g(x))\,dx$$

$$= \int f(x)\cdot g'(x)\,dx + \int f'(x)\cdot g(x)\,dx.$$

$$\therefore \int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx.$$

(2). Since, $0 < a < 2a$

$$\int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \text{ ---- } (1)$$

Taking $x = 2a - t$ in $I = \int_a^{2a} f(x)\,dx$.

$\therefore dx = -dt.$

17.

$\therefore$ as $x \to a \Rightarrow t \to a$, and

$x \to 2a \Rightarrow t \to 0$.

$$\therefore \quad I = \int_a^{2a} f(x)\,dx$$

$$= \int_a^0 f(2a-t)(-dt)$$

$$= -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx.$$

$\therefore$ From ①, we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$$

(B) Answer any two:

(1) $I = \int \dfrac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}\,dx$.

let $a^2\sin^2 x + b^2\cos^2 x = t$.

$\therefore \left[a^2 \cdot 2\sin x\cos x + b^2 \cdot 2\cos x(-\sin x)\right] dx = dt$

$\therefore \quad \sin 2x\,(a^2 - b^2)\,dx = dt$

$\therefore \quad \sin 2x\,dx = \dfrac{dt}{a^2 - b^2}$

$$\therefore \quad I = \frac{1}{a^2-b^2}\int \frac{dt}{t}$$

$$= \frac{1}{a^2-b^2}\log|t| + c$$

$$= \frac{1}{a^2-b^2}\log\left|a^2\sin^2 x + b^2\cos^2 x\right| + C.$$

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(2) $I = \int \log\left(x + \sqrt{x^2+a^2}\right) dx.$

$$\frac{d}{dx}\left[\log\left(x + \sqrt{x^2+a^2}\right)\right]$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1 + \frac{1}{2\sqrt{x^2+a^2}} \cdot 2x\right]$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+a^2}}} \cdot \left[\frac{\cancel{\sqrt{x^2+a^2}+x}}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{1}{\sqrt{x^2+a^2}}$$

$$\therefore I = \int \log\left(x + \sqrt{x^2+a^2}\right) \cdot 1 \, dx$$

$$= \log\left(x + \sqrt{x^2+a^2}\right) \cdot \int 1\, dx - \int \left[\frac{d}{dx}\left(\log\left(x + \sqrt{x^2+a^2}\right)\right) \cdot \int 1\, dx\right] dx$$

$$= x \log\left(x + \sqrt{x^2+a^2}\right) - \int\left(\frac{1}{\sqrt{x^2+a^2}} \cdot x\right) dx$$

$$= x \cdot \log\left(x + \sqrt{x^2+a^2}\right) - \frac{1}{2}\int \frac{2x}{\sqrt{x^2+a^2}}\, dx$$

$$= x \log\left(x + \sqrt{x^2+a^2}\right) - \frac{1}{2} \cdot 2\sqrt{x^2+a^2} + c$$

$$= x \log\left(x + \sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2} + c.$$

(3) $\int \frac{x^2}{(2x^2+1)(x^2-1)}\, dx.$

$$= \frac{1}{3}\int \frac{3x^2}{(2x^2+1)(x^2-1)}\, dx$$

$$= \frac{1}{3}\int \frac{(2x^2+1)+(x^2-1)}{(2x^2+1)(x^2-1)}\, dx$$

$$= \frac{1}{3}\int \frac{1}{x^2-1}\, dx + \int \frac{1}{2x^2+1}\, dx \Big]$$

$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2} \cdot \frac{1}{1/\sqrt{2}} \tan^{-1}\left(\frac{x}{1/\sqrt{2}}\right)\right] + c$$

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$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x)\right] + c$$

$$= \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{3\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C.$$

(c) Evaluate any two:

(1) $\int_1^2 6^x \, dx$.

Here, $f(x) = 6^x$ is increasing and continuous on $[1, 2]$. Divide $[1, 2]$ into n equal sub-interval. length of each subinterval is h.

$\therefore \quad h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$\therefore \quad nh = 1.$

$$\therefore \int_a^b f(x)\,dx = \lim_{h \to 0} h \cdot \sum_{i=1}^{n} f(a + ih)$$

$$\therefore \int_1^2 6^x dx = \lim_{h \to 0} h \cdot \sum_{i=1}^{n} 6^{1+ih}$$

$$= \lim_{h \to 0} h \cdot \sum_{i=1}^{n} 6 \cdot 6^{ih}$$

$$= \lim_{h \to 0} 6h \cdot \sum_{i=1}^{n} 6^{ih}$$

$$= \lim_{h \to 0} 6h \left[6^h + 6^{2h} + 6^{3h} + \cdots + 6^{nh}\right]$$

$$= \lim_{h \to 0} 6h \cdot \frac{6^h(6^{nh}-1)}{(6^h - 1)}$$

$$= \frac{6 \cdot \lim_{h \to 0} 6^h \cdot \lim_{h \to 0} (6^{nh} - 1)}{\lim_{h \to 0}\left(\frac{6^h - 1}{h}\right)}$$

$$= \frac{6 \cdot 6^0 \cdot (6-1)}{\log_e 6}$$

$$= \frac{30}{\log_e 6} = 30 \cdot \log_6 e$$

20.

(2) $$\int_0^{\pi/2} \frac{\cos x \, dx}{(1+\sin x)(2+\sin x)}$$

let $1+\sin x = t$

$\therefore \cos x \, dx = dt$

When $x = 0 \Rightarrow t = 1$

$x = \pi/2 \Rightarrow t = 2.$

$$\therefore I = \int_1^2 \frac{dt}{t(t+1)}$$

$$= \int_1^2 \frac{(t+1)-t}{t(t+1)} dt$$

$$= \int_1^2 \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

$$= \left[\log t - \log(t+1) \right]_1^2$$

$$= \left[\log \frac{t}{t+1} \right]_1^2$$

$$= \log \frac{2}{3} - \log \frac{1}{2}$$

$$= \log 4/3.$$

(3) $$\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}.$$

$$\therefore I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$$

$$= \int_0^{\pi} \frac{x \sin x / \cos x}{\frac{1}{\cos x} + \cos x} dx$$

$$= \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx.$$

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$$= \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x} - I$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x}$$

let $\cos x = t$

$\therefore \; -\sin x \, dx = dt$

$\therefore \; \sin x \, dx = -dt$

When $x = 0 \Rightarrow t = 1$

$x = \pi \Rightarrow t = -1$

$$\therefore \quad 2I = \pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$= \pi \int_{-1}^{1} \frac{1}{1+t^2} \, dt$$

$$= 2\pi \int_0^1 \frac{1}{1+t^2} \, dt \qquad (\because \text{ function is even}).$$

$$= 2\pi \left[\tan^{-1} t \right]_0^1$$

$$= 2\pi \cdot \frac{\pi}{4}$$

$$= \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}$$

(D) (1) $$I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} \, dx$$

$$= \int \frac{4e^x + 6 \cdot \frac{1}{e^x}}{9e^x - 4 \cdot \frac{1}{e^x}} \, dx$$

$$= \int \frac{4 \cdot e^{2x} + 6}{9 \cdot e^{2x} - 4} \cdot dx$$

let $e^{2x} = t$

$\therefore \; e^{2x} \cdot 2 \, dx = dt$

$\therefore \; dx = \frac{1}{2} \cdot \frac{dt}{t}$

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$$\therefore\ I = \int \frac{4t+6}{(9t-4)} \frac{dt}{2t}$$

$$= \int \frac{2t+3}{t(9t-4)}\, dt$$

$$\frac{2t+3}{t(9t-4)} = \frac{A}{t} + \frac{B}{9t-4}$$

$$2t+3 = A(9t-4) + Bt$$

For $t = 0$, $3 = -4A$ $\therefore A = -3/4$

$t = \frac{4}{9}$, $2 \cdot \frac{4}{9} + 3 = B \cdot \frac{4}{9}$

$$\therefore \frac{8+27}{9} = \frac{4B}{9}$$

$$\therefore 4B = 35$$

$$\therefore B = 35/4.$$

$$\therefore\ I = -\frac{3}{4}\int \frac{dt}{t} + \frac{35}{4}\int \frac{dt}{9t-4}$$

$$= -\frac{3}{4}\log|t| + \frac{35}{4}\frac{\log|9t-4|}{9} + C$$

$$= \frac{35}{36}\log|9e^{2x}-4| - \frac{3}{4}\log|e^{2x}| + C$$

$$= \frac{35}{36}\log|9e^{2x}-4| - \frac{3}{2}x + C.$$

(2) $\int (f(x) + f'(x))e^x\,dx$

$$= \int f(x)\cdot e^x\,dx + \int f'(x)\cdot e^x\,dx$$

$$= f(x)\int e^x dx - \int\left(\frac{d}{dx}f(x)\cdot\int e^x dx\right)dx + \int f'(x)\cdot e^x dx$$

$$= f(x)\cdot e^x - \cancel{\int f'(x)\cdot e^x dx} + \cancel{\int f'(x)\cdot e^x dx}$$

$$= f(x)\cdot e^x + C.$$

Q.4 (A) Answer any two :

(1) $e^{y/x}\left(x\cdot\frac{dy}{dx} - y\right) + x = 0.$

$\therefore\ e^{y/x}\, x\cdot\frac{dy}{dx} - y\cdot e^{y/x} + x = 0.$

$\therefore\ e^{y/x}\cdot x\cdot\frac{dy}{dx} = y\cdot e^{y/x} - x$

$\therefore\ \frac{dy}{dx} = \frac{y\cdot e^{y/x} - x}{x\cdot e^{y/x}}$

$\therefore\ \frac{dy}{dx} = \frac{y}{x} - \frac{1}{e^{y/x}}$

$\therefore\ \frac{dy}{dx} = f\left(y/x\right)$

This is homogeneous differential equation.

let $y/x = v.$

$\therefore\ y = vx.$

$\therefore\ \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}.$

$\therefore\ v + x\cdot\frac{dv}{dx} = v - \frac{1}{e^{v}}$

$\therefore\ x\frac{dv}{dx} = -\frac{1}{e^{v}}$

$\therefore\ e^{v}dv + \frac{dx}{x} = 0.$

Integrating termwise,

$\int e^{v}dv + \int\frac{dx}{x} = \log|c|$

$\therefore\ e^{v} + \log|x| = \log|c|.$

$\therefore\ e^{y/x} = \log\left|\frac{c}{x}\right|$

This is general solution. and c is arbitrary constant.

24.

(2) $\frac{dy}{dx} = (9x+y+1)^2$

let $9x+y+1 = t$

$\therefore 9 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dy}{dx} = \frac{dt}{dx} - 9$

$\therefore \frac{dt}{dx} - 9 = t^2$ ($\because$ Substituting in equation).

$\therefore \frac{dt}{dx} = t^2 + 9.$

$\therefore \frac{dt}{t^2+9} = dx.$

Integrating termwise

$\int dx = \int \frac{dt}{t^2+3^2}$

$\therefore x = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c'$

$\therefore 3x = \tan^{-1}\left(\frac{t}{3}\right) + 3c'$

$\therefore 3x - 3c' = \tan^{-1}\left(\frac{t}{3}\right)$

$\therefore \tan^{-1}\left(\frac{t}{3}\right) = 3x - c$ $(3c' = c)$

$\therefore \frac{t}{3} = \tan(3x-c)$

$\therefore t = 3\tan(3x-c)$

$\therefore 9x+y+t = 3\tan(3x-c).$

This is general solution and c is arbitrary Constant.

(3) $\frac{dy}{dx} + y = e^x$

This is linear differential equation in the form of $\frac{dy}{dx} + Py = Q$.

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$P = 1 \quad , \quad Q = e^{x}$

Multiplying both sides by $e^{\int P\,dx} = e^{\int dx} = e^{x}$

$\therefore$ Equation is $e^{x}.\frac{dy}{dx} + e^{x}.y = e^{2x}$

Now, $\frac{d}{dx}(y.e^{x}) = y.e^{x} + e^{x}.\frac{dy}{dx}$

$\therefore \frac{d}{dx}(e^{x}.y) = e^{2x}$

$\therefore e^{x}.y = \int e^{2x}dx + c'$

$\therefore e^{x}.y = \frac{e^{2x}}{2} + c'$

$\therefore 2.e^{x}.y = e^{2x} + 2c'$

$\therefore 2e^{x}.y = e^{2x} + c$

or $2y = e^{x} + c.e^{-x}$

This is the general solution.

When $x = 0$, $y = 1$.

$\therefore 2 = e^{0} + ce^{0}$

$\therefore 2 = 1 + c$

$\therefore c = 1$.

$\therefore 2y = e^{x} + e^{-x}$ is particular solution.

(4). Let the mass of the radium be m.

The rate of decay of radium be $\frac{dm}{dt}$

$\therefore \frac{dm}{dt} \propto m$

$\therefore \frac{dm}{dt} = -km \quad , \quad k > 0$.

$\therefore \frac{dm}{m} = -k\,dt$

$\therefore \int \frac{dm}{m} = -k\int dt + \log c$.

$\therefore \log m = -kt + \log c$.

$\therefore \log m = \log e^{-kt} + \log c$.

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$\therefore \log m = \log c \cdot e^{-kt}$

$\therefore \boxed{m = c \cdot e^{-kt}}$

When $t = 0$, $m = m_0$

$\therefore m_0 = c \cdot e^0$

$\therefore c = m_0$

$\therefore \boxed{m = m_0 \cdot e^{-kt}}$

When $t = t_1$, $m = m_1$

$t = t_2$, $m = m_2$

$\therefore m_1 = m_0 \cdot e^{-kt_1}$

$m_2 = m_0 \, e^{-kt_2}$

$\therefore \frac{m_1}{m_2} = \frac{e^{-kt_1}}{e^{-kt_2}}$

$\therefore \frac{m_1}{m_2} = e^{k(t_2 - t_1)}$.

$\therefore k(t_2 - t_1) = \log_e \frac{m_1}{m_2}$

$\therefore k = \frac{1}{t_2 - t_1} \cdot \log_e \frac{m_1}{m_2}$

When $m = \frac{m_0}{2}$

$\frac{m_0}{2} = m_0 \cdot e^{-kt}$

$\therefore \frac{1}{2} = e^{-kt}$

$\therefore 2 = e^{kt}$

$\therefore kt = \log_e 2$

$\therefore t = \frac{1}{k} \log_e 2$

$= \frac{1}{\frac{1}{t_2 - t_1} \cdot \log \frac{m_1}{m_2}} \cdot \log_e 2$

$$\therefore t = \frac{(t_2 - t_1)\log_e 2}{\log_e \frac{m_1}{m_2}}$$

(B) Attempt any two :

27. Find the area of the region bounded by $x = 3y^2 - 9$, y-axis and lines $y = 0$ and $y = 1$.

$\therefore$ At y-axis, $x = 0$.

$\therefore 3y^2 - 9 = 0$

$\therefore 3y^2 = 9$

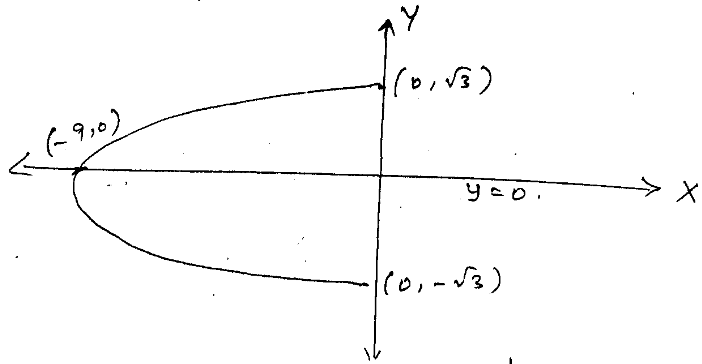
$\therefore y^2 = 3$

$\therefore y = \pm\sqrt{3}$

$\therefore$ The points are $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

When $y = 0 \Rightarrow x = -9$.

$\therefore$ The point is $(-9, 0)$.



$A = |I|$, where $I = \int_a^b x\,dy$

$= \int_0^1 (3y^2 - 9)\,dy$

$= \left[3 \cdot \int y^2 dy - 9\int dy\right]_0^1$

$= \left[3 \cdot \frac{y^3}{3} - 9y\right]_0^1$

$= 1 - 9$

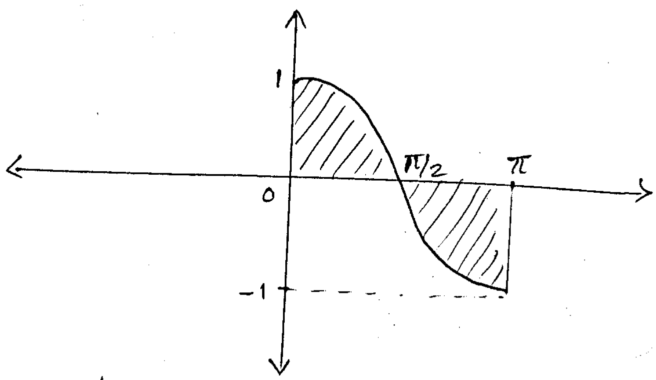
$= -8$

28.

$\therefore A = |-8|$

$= 8$

(2).



$$A = \left| \int_a^b y\, dx \right|$$

$$= \left| \int_0^{\pi/2} \cos x\, dx \right| + \left| \int_{\pi/2}^{\pi} \cos x\, dx \right|$$

$$= \left| \left[\sin x \right]_0^{\pi/2} \right| + \left| \left[\sin x \right]_{\pi/2}^{\pi} \right|$$

$$= \left| \left(\sin \pi/2 - \sin 0 \right) \right| + \left| \left(\sin \pi - \sin \pi/2 \right) \right|$$

$$= |1| + |-1|$$

$$= 1 + 1$$

$$= 2.$$

(3) Solving the equations

$y = x^2 + 1$, $y = 2x + 1$.

$\therefore x^2 + 1 = 2x + 1$

$\therefore x^2 - 2x = 0$

$\therefore x(x-2) = 0$

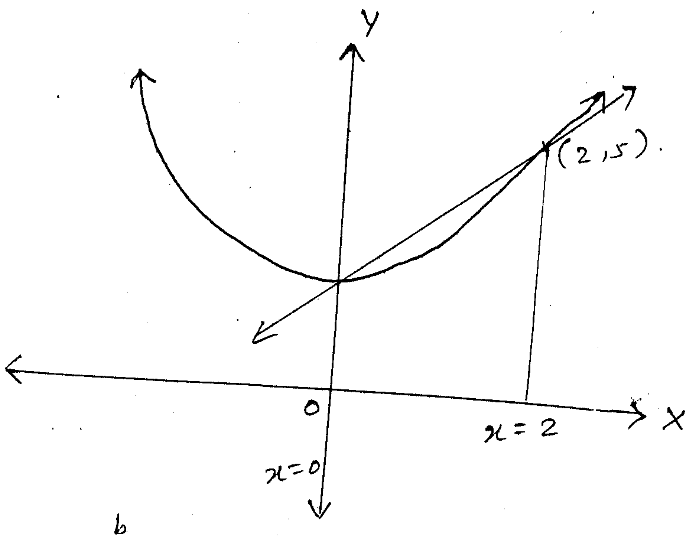
$\therefore x = 0$ or $x = 2$.

When $x = 0 \Rightarrow y = 1$

$x = 2 \Rightarrow y = 5$

$\therefore$ The intersection points are $(0,1)$ & $(2,5)$.

29.



$$V = \pi \int_a^b \left[(f_1(x))^2 - (f_2(x))^2 \right] dx$$

$$= \pi \int_0^2 \left[(2x+1)^2 - (x^2+1)^2 \right] dx$$

$$= \pi \int_0^2 (4x^2 + 4x + 1 - x^4 - 2x^2 - 1)\, dx$$

$$= \pi \int_0^2 (2x^2 + 4x - x^4)\, dx$$

$$= \pi \left[2\int x^2 dx + 4\int x\, dx - \int x^4 dx \right]_0^2$$

$$= \pi \left[2 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} - \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\frac{2x^3}{3} + 2x^2 - \frac{x^5}{5} \right]_0^2$$

$$= \pi \left[\frac{16}{3} + 8 - \frac{32}{5} \right]$$

$$= \pi \left[\frac{80 + 120 - 96}{15} \right]$$

$$= \frac{104\pi}{15}$$

30.

(c). Let white colour is denoted by W and red colour is denoted by R. Let five distinct balls are W_1, W_2, R_1, R_2, R_3.
Two balls are selected from box which can be done is $\binom{5}{2}$ ways = 10 ways.

$\therefore$ U be the sample space associated with the experiment of selecting two balls.

$\therefore U = \{W_1W_2, W_1R_1, W_1R_2, W_1R_3, W_2R_1, W_2R_2, W_2R_3, R_1R_2, R_2R_3, R_1R_3\}$.

$\therefore X(U)$ = Number of white balls in U.

u	P(U)	X(u) = x	p(x) =
W_1W_2	1/10	2	1/10
W_1R_1	1/10	1	$\frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10} = \frac{6}{10}$
W_1R_2	1/10		
W_1R_3	1/10		
W_2R_1	1/10		
W_2R_2	1/10		
W_2R_3	1/10		
R_1R_2	1/10	0	$\frac{1}{10}+\frac{1}{10}+\frac{1}{10} = \frac{3}{10}$
R_1R_3	1/10		
R_2R_3	1/10		

The probability distribution of a random variable x.

X = x	0	1	2
p(x)	3/10	6/10	1/10

(2) Mean $= np = 4$

Variance $= npq = 2$

$\therefore \frac{npq}{np} = \frac{2}{4} = \frac{1}{2}$

$\therefore q = \frac{1}{2} \quad \therefore p = \frac{1}{2}$

31.

& $np = 4$ $\therefore n = 8$.

$\therefore$ Binomial distribution of a random variable x.

$\therefore P(X=x) = \binom{n}{x} \cdot p^x \cdot q^{n-x}$

$\therefore P(X=x) = \binom{8}{x} \cdot \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x}$

$= \binom{8}{x} \left(\frac{1}{2}\right)^8$

$P(1 \le x \le 3) = P(X=1) + P(X=2) + P(X=3)$

$= \binom{8}{1}\left(\frac{1}{2}\right)^8 + \binom{8}{2}\left(\frac{1}{2}\right)^8 + \binom{8}{3}\left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 \left[\binom{8}{1} + \binom{8}{2} + \binom{8}{3}\right]$

$= \left(\frac{1}{2}\right)^8 [8 + 28 + 56]$

$= \frac{1}{2^8} \times 92 = \frac{92}{256}$

$= \frac{23}{64}$

(3). $p(x) = \binom{6}{x} \cdot p^x \cdot q^{6-x}$

$3 \cdot p(2) = 2 \cdot p(3)$.

$\therefore 3\binom{6}{2} p^2 q^4 = 2 \cdot \binom{6}{3} p^3 q^3$

$\therefore 3 \cdot \frac{6 \times 5}{2 \times 1} \cdot p^2 q^4 = 2 \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \cdot p^3 q^3$

$\therefore \frac{3}{2} p^2 q^4 = \frac{4}{3} \cdot p^3 q^3$

$\therefore \frac{3}{2} q = \frac{4}{3} p$.

$\therefore \frac{3}{2}(1-p) = \frac{4p}{3}$

$\therefore 9(1-p) = 8p$.

$\therefore 9 - 9p = 8p$

$\therefore 17p = 9$

$\therefore p = \frac{9}{17}$.

32

D (1) $\int (x^4 - 3x^3 + 7x^2 - 2x - 1)\, e^x dx$

$= \int [(x^4 - 7x^3 + 28x^2 - 58x + 59) + (4x^3 - 21x^2 + 56x - 58)]\, e^x dx$

$f(x) = x^4 - 7x^3 + 28x^2 - 58x + 59$

$f'(x) = 4x^3 - 21x^2 + 56x - 58$

$\therefore\ I = (x^4 - 7x^3 + 28x^2 - 58x + 59)e^x + c.$

(2) Differential equations are used in many faculties like Biology, atomic physics, Dynamics, economics, medical science for solving exponential and practical problems.

(3). Linear differential equation is

$\frac{dy}{dx} + py = Q$. (p is constant and Q is a function of x).

$\therefore$ Multiply both sides by $e^{\int p\,dx} = e^{px}$.

$\therefore\ e^{px} \cdot \frac{dy}{dx} + p \cdot y \cdot e^{px} = Q \cdot e^{px}$.

Now, $\frac{d}{dx}(y \cdot e^{px}) = y \cdot e^{px} \cdot p + e^{px} \cdot \frac{dy}{dx}$

$\therefore\ \frac{d}{dx}(y \cdot e^{px}) = Q \cdot e^{px}$.

Integrating termwise.

$\therefore\ y \cdot e^{px} = \int Q e^{px} dx + c$

This is the general solution of equation. and c is arbitrary constant.

33

Q.5.

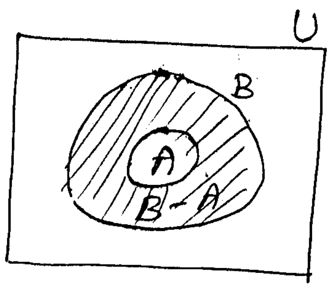
(A).(1).

(i) Exhaustive events: Let A and B be events. If $A \cup B = U$ then A and B are called exhaustive events.

(ii) Elementary events: let U be the finite sample space and let $U = \{x_1, x_2, \ldots, x_n\}$. Singleton subsets $\{x_i\}$ of U for $i = 1, 2, \ldots, n$ are called elementary events.

OR

(1) Here, $A \subset B$ is given. According to venn diagram, events A and $B - A$ are mutually exclusive events.



$A \subset B$.

Also, $A \cup (B - A) = B$.

$\therefore \quad P(A \cup (B - A)) = P(B)$.

$\therefore \quad P(A) + P(B - A) = P(B)$

$\therefore \quad P(A - B) = P(B) - P(A)$.

Also, for any event, $B - A$, $P(B - A) \geq 0$.

$\therefore \quad P(B) - P(A) \geq 0$

$\therefore \quad P(B) \geq P(A)$

or $\quad P(A) \leq P(B)$.

(2) (i) $P(A \cap B') = P(A) - P(A \cap B)$

$= P(A) - P(A) \cdot P(B)$ ($\because$ A and B are independent)

$= P(A)(1 - P(B))$

$= P(A) \cdot P(B')$.

$\therefore$ A and B' are Independent events.

34.

(ii) $P(B \cap A') = P(B) - P(A \cap B)$

$= P(B) - P(A) \cdot P(B)$ ($\because$ A & B are independent)

$= P(B)(1 - P(A))$

$= P(B) \cdot P(A')$.

$\therefore$ B and A' are independent events.

(B) Attempt any two:

(i). $U = \{1, 2, \ldots, 4000\}$.

let A = Event that the number is multiple of 10.

B = Event that the number is multiple of 25.

$\therefore A \cap B$ = Event that the number is multiple of 50.

$\therefore A = \{10, 20, 30, \ldots, 4000\}$.

$B = \{25, 50, 75, \ldots, 4000\}$

$\therefore A \cap B = \{50, 100, \ldots, 4000\}$.

$\therefore P(A) = \frac{400}{4000} = \frac{1}{10}$

$P(B) = \frac{160}{4000} = \frac{1}{25}$

$P(A \cap B) = \frac{80}{4000} = \frac{1}{50}$.

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{10} + \frac{1}{25} - \frac{1}{50}$

$= \frac{5 + 2 - 1}{50}$

$= \frac{6}{50}$

$= \frac{3}{25}$

35

(2). let B_1 = Event that the selected person is a man

B_2 = Event that the selected person is a woman.

$\therefore P(B_1) = P(B_2) = 1/2$

A = Event that the selected person is suffering from colour blindness.

$\therefore P(A/B_1) = \frac{3}{100}$, $P(A/B_2) = \frac{3}{1000}$.

$$\therefore P(A) = P(B_1)\cdot P(A/B_1) + P(B_2)\cdot P(A/B_2)$$

$$= \frac{1}{2}\times\frac{3}{100} + \frac{1}{2}\times\frac{3}{1000}$$

$$= \frac{3}{200} + \frac{3}{2000}$$

$$= \frac{30+3}{2000}$$

$$= \frac{33}{2000}$$

$$\therefore P(B_2/A) = \frac{P(B_2)\cdot P(A/B_2)}{P(B_2)}$$

$$= \frac{\frac{3}{1000}\times\frac{1}{2}}{33/2000}$$

$$= \frac{1}{11}.$$

(3). let G_1 and G_2 be the event that selecting green ball first and second time respectively.

let R_1 and R_2 be the event that the selecting red ball first and second time respectively.

36.

P(Event that the first ball is green and second ball is red).

$= P(G_1 \cap R_1)$

$= P(G_1) \cdot P(R_1)$

$= \dfrac{\binom{g}{1}}{\binom{r+g}{1}} \times \dfrac{\binom{r}{1}}{\binom{r+g-1}{1}}$

$= \dfrac{g \cdot r}{(r+g)(r+g-1)}$

(ii) P(Event that the first ball is green and the second ball is also green)

$= P(G_1 \cap G_2)$

$= P(G_1) \cdot P(G_2)$

$= \dfrac{\binom{g}{1}}{\binom{r+g}{1}} \times \dfrac{\binom{g-1}{1}}{\binom{r+g-1}{1}}$

$= \dfrac{g(g-1)}{(r+g)(r+g-1)}$

(c) let G_1, G_2, G_3 be the event that selecting girl from 1^{st}, 2^{nd}, 3^{rd} family respectively.

let B_1, B_2, B_3 be the event that selecting boy from 1^{st}, 2^{nd}, 3^{rd} family respectively.

$\therefore P(G_1) = 3/4$, $P(G_2) = 2/4$, $P(G_3) = 1/4$.
$P(B_1) = 1/4$, $P(B_2) = 2/4$, $P(B_3) = 3/4$.

P(Event that selecting 2 girls and one boy)

$= P((G_1 \cap G_2 \cap B_3) \cup (G_1 \cap B_2 \cap G_3) \cup (B_1 \cap G_2 \cap G_3))$

$= P(G_1) \cdot P(G_2) \cdot P(B_3) + P(G_1) \cdot P(B_2) \cdot P(G_3) + P(B_1) \cdot P(G_2) \cdot P(G_3)$.

$= \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{2}{4} \cdot \frac{1}{4}$

$= \frac{18}{64} + \frac{6}{64} + \frac{2}{64}$

$= \frac{26}{64}$

$= \frac{13}{32}$.

(2) Random experiment of selecting 3 numbers from 1 to 20.

$\therefore n = \binom{20}{3}$.

let A = Randomly selected three numbers are consecutive numbers.

$\therefore A = \{(1,2,3), (2,3,4), \ldots, (18,19,20)\}$.

$\therefore n(A) = 18$.

$\therefore P(A) = \frac{r}{n} = \frac{18}{\binom{20}{3}} = \frac{\cancel{18} \times 3 \times 2 \times 1}{20 \times 19 \times \cancel{18}}$

$= \frac{3}{196}$.

(3). Here, $P(A' \cap B) = 0.1$, $P(A \cap B') = 0.4$, $P(A' \cup B') = 0.6$.

$\therefore P(A' \cup B') = 0.6$

$\therefore 1 - P(A \cap B) = 0.6$

$\therefore P(A \cap B) = 0.4$.

As, $P(A' \cap B) = 0.1$

$\therefore P(B) - P(A \cap B) = 0.1$

$\therefore P(B) - 0.4 = 0.1$

$\therefore \boxed{P(B) = 0.5}$

and, $P(A \cap B') = 0.4$

$\therefore P(A) - P(A \cap B) = 0.4$

$\therefore P(A) - 0.4 = 0.4$

$\therefore \boxed{P(A) = 0.8}$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.8 + 0.5 - 0.4$

$\therefore P(A \cup B) = 0.9$

(D) (1) Names of the important language used in computers are

(i) BASIC, (ii) COBOL (iii) FORTRAN

(iv) PASCAL

(2) $(1101.101)_2$

$= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 \times 1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}$

$= 8 + 4 + 0 + 1 + 0.5000 + 0.250 + 0.125$

$= (13.875)_{10}$

$(1101.101)_2 = (\underline{001}\,\underline{101}.\underline{101})_2$

$= (15.5)_8$.

because $001 = 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1$

$101 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$

$101 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 5$.