# CHAPTER 14

# HERON'S FORMULA

## 14.1 Introduction

In the previous classes, we have studied about the figures of different shapes such as a triangle, a square, a rectangle, a rhombus, a trapezium etc. Moreover, we had found out the areas of regions enclosed by the figures and also calculated the perimeters of them. For example, if we want to find out the perimeter of any floor of a room of our school or home, it is obvious that we walk around the boundary of that room. The total distance covered by us is considered as perimeter of that room and the floor of that room will have an area also.

So if the floor of our room is rectangular and its length is $l$ and breath is $b$, then total distance covered will be $2(l+b)$ i.e. its perimeter and its area is $lb$.

How can we find the area of a triangle ? We know the following result about area.

**Area $= \frac{1}{2} \times$ base $\times$ altitude** **(i)**

For a right angled triangle we can use the above formula directly because an altitude from the vertex to the base of the triangle will be a side of the triangle. For example, in the right angled $\Delta$ ABC, $m\angle B = 90$, AB $= 3$ *cm*, BC $= 4$ *cm*, length of the hypotenuse AC $= 5$ *cm*. Then the area of the triangle is given by $\frac{1}{2} \times AB \times BC$ where AB is the altitude and BC is the base of the triangle.

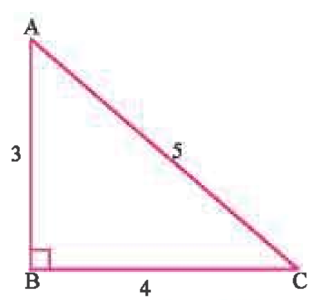


Figure 14.1

Area $= \frac{1}{2} \times 4 \times 3 = 6\ cm^2$

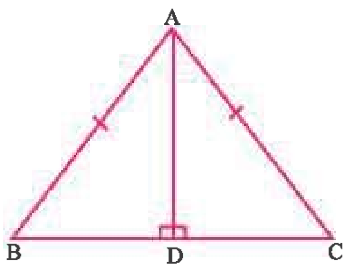


Figure 14.2

Let us find out the area of an isosceles triangle with the help of the above formula. In $\Delta$ ABC, let AB = AC. Now draw the perpendicular from the vertex A to the base $\overline{BC}$ which intersects $\overline{BC}$ at D. Thus, $\Delta$*ABC is divided into two triangular regions, $\Delta$*ABD and $\Delta$*ACD.

$m\angle ADB = m\angle ADC = 90$

Now if AB = 5 *cm*, then AC is also 5 *cm* and let BC = 6 *cm*. Altitude from A divides $\overline{BC}$ in two congruent line-segments $\overline{BD}$ and $\overline{DC}$. Thus BD + DC = BC, so that BD = DC = 3 *cm* (figure 14.2)

Now, apply Pythagoras' theorem to the right angled $\Delta$ ADB

$AB^2 = BD^2 + AD^2$

$\therefore\ 5^2 = (3)^2 + AD^2$

$\therefore\ 25 - 9 = AD^2$

$\therefore\ AD^2 = 16$

$\therefore$ AD = 4 *cm* = length of the altitude

$\therefore$ By (i), area of the isosceles $\Delta$ ABC = $\frac{1}{2} \times 6 \times 4 = 12\ cm^2$

Similarly, we want to find the area of an equilateral $\Delta$ ABC, where the length of each side is 12 *cm*. For this triangle, if we draw a perpendicular from the vertex A to the base $\overline{BC}$ which intersects $\overline{BC}$ at D, then $\overline{AD}$ is an altitude of $\Delta$ABC. Here D is the midpoint of $\overline{BC}$.

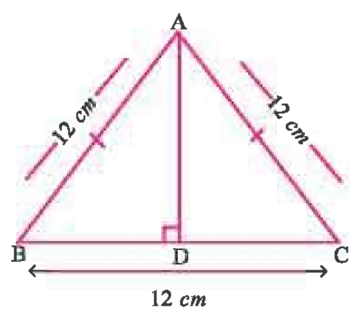


Figure 14.3

Thus, BD = DC = 6 *cm* (figure 14.3)

For right angled $\Delta$ADB, $AB^2 = BD^2 + AD^2$

$\therefore\ (12)^2 = AD^2 + (6)^2$

$\therefore\ AD^2 = 144 - 36$

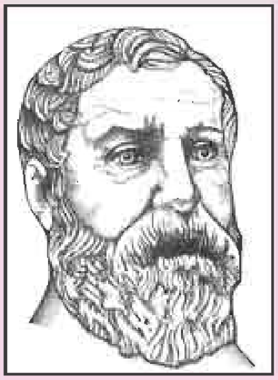
$\therefore\ AD^2 = 108$

$\therefore\ AD = 6\sqrt{3}\ cm$

$\therefore$ The area of equilateral $\Delta$ ABC is given by, $\frac{1}{2} \times AD \times BC = \frac{1}{2} \times 6\sqrt{3} \times 12$

$\therefore$ The area of $\Delta ABC = 36\sqrt{3}\ cm^2$

## 14.2 Heron's Formula



Heron (10AD - 75 AD)

Heron was born in about 10 A.D. possibly in Alexandria in Egypt. He worked in applied mathematics. His work on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields. His geometrical works deal largely with problems on mensuration written in three books. Book I deals with the area of squares, rectangles, triangles, trapezoids (trapezia), various other specialised quadrilaterals, the regular polygons, circles, surfaces of cylinders, cones, spheres etc. In this book, Heron has derived the famous formula for the area of a triangle in terms of its three sides.

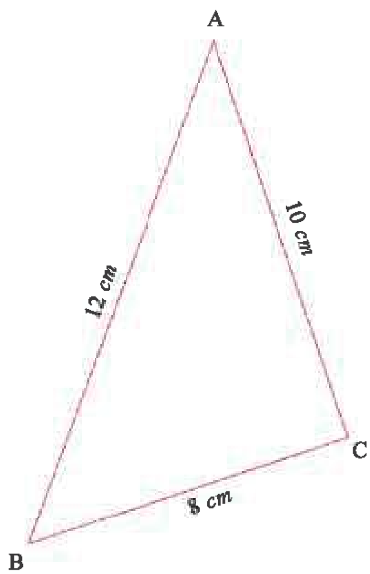



Figure 14.4

For an isosceles, equilateral and right angled triangle, we can draw the perpendiculars from the vertex to the base and we can find their lengths. Then we can find the area of the triangle by using the formula $\frac{1}{2} \times$ base $\times$ altitude. But if we have a scalene triangle, then we do not have any clue to find the length of an altitude (i.e. perpendicular from a vertex to the base of the triangle).

For an example, in $\Delta$ ABC, Let AB = 12 *cm*, BC = 8 *cm* and AC = 10 *cm*. Now there is a problem as to how can we calculate the area of this triangle ? For this, a formula is given by Heron, which is known as **Heron's formula**. It is as follows :

**Area of a triangle** $= \sqrt{s(s-a)(s-b)(s-c)}$ **(ii)**

Here *a*, *b*, *c* are the lengths of the sides of the triangle and *s* is semiperimeter of the triangle.

Thus, perimeter $= a + b + c = 2s$

$\therefore s = \frac{a+b+c}{2}$

So, if the length of the altitude is not given and it is not easy to find it, then this formula (ii) will be helpful to find the area of the triangle. So for the above example,

$s = \frac{12+10+8}{2} = 15\ cm$

Area of $\Delta$ ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{15\,(15-12)\,(15-10)\,(15-8)}$

$= \sqrt{15(3)(5)(7)} = 15\sqrt{7}\ cm^2$

Let us solve following examples to understand the application of Heron's formula.

**Example 1 :** Find the area of the triangle whose sides have lengths 15, 15, 12 *cm*.

**Solution :** Here, $s = \frac{a+b+c}{2} = \frac{15+15+12}{2} = \frac{42}{2} = 21\ cm$

$\therefore$ The area of $\Delta$ ABC $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

$= \sqrt{21\,(21-15)\,(21-15)\,(21-12)}$

$= \sqrt{21 \times 6 \times 6 \times 9}$

$= 18\sqrt{21}\ cm^2$

(Do you have any other alternative method ?)

**Example 2 :** The lengths of the sides of a triangular park are in proportion 3 : 5 : 7 and its perimeter is 450 metre, then find out the area of this park. Also find the cost of fencing it with barbed wire at the rate of ₹ 25 per metre by leaving a space of 5 metre wide for a gate on all the sides.

**Solution :** The sides are in the proportion 3 : 5 : 7. Suppose the lengths of the sides of the triangular park are $3x$, $5x$ and $7x$. ($x > 0$).

Now, perimeter of triangular park = 450 metre

$\therefore\ 3x + 5x + 7x = 450$

$\therefore\ 15x = 450$

$\therefore\ x = 30$ metre

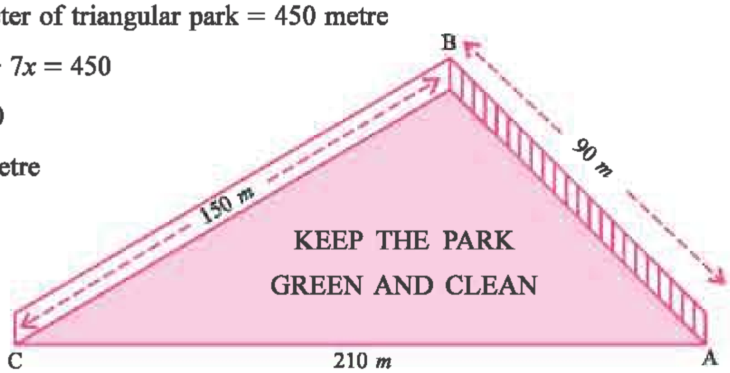



Figure 14.5

Thus, for $\Delta$ ABC, AB = $c$ = $3x$ metre = 3(30) = 90 metre

BC = $a$ = $5x$ metre = 5(30) = 150 metre

AC = $b$ = $7x$ metre = 7(30) = 210 metre

Now, $s = \frac{a+b+c}{2} = \frac{90+150+210}{2} = \frac{450}{2} = 225$ metre

$\therefore$ The area of $\Delta$ ABC $= \sqrt{225(225-90)(225-150)(225-210)}$

$= \sqrt{225(135)(75)(15)}$

$= \sqrt{15\times15\times15\times9\times25\times3\times15}$

$= \sqrt{(15)^4\times(5)^2\times(3)^2\times3}$

$= (15)^2\times5\times3\times\sqrt{3}$

$= 3375\sqrt{3}\ m^2$

Now, for the fencing, 5 metre space is left on each side of the triangular park. Then total space left will be 5 × 3 = 15 $m$. Hence the total length for the fencing = length of the wire needed for fencing = Permeter of the triangular park – length of the gates

= 450 metre − 15 metre = 435 metre

$\therefore$ Total cost of fencing = 435 × 25

= ₹ 10875

**Example 3 :** Find the area of the triangle $\Delta$ABC where AB = 5 $cm$, BC = 8 $cm$ and AC = 9 $cm$ . Find the length of the perpendicular drawn from A to $\overline{BC}$

**Solution :** Here, $s = \frac{a+b+c}{2} = \frac{5+8+9}{2} = 11\ cm$

$\therefore$ The area of $\Delta$ ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{11(11-8)(11-9)(11-5)}$

$= \sqrt{11\times3\times2\times6}$

$= \sqrt{11\times(6)^2}$

$= 6\sqrt{11}\ cm^2$

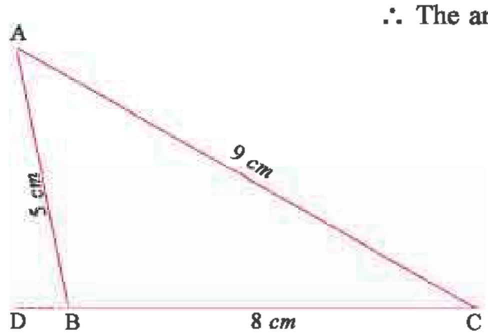



Figure 14.6

Here, $\overline{AD} \perp \overline{BC}$ (see figure 14.6)

Now we have, area of $\Delta$ABC

$= \frac{1}{2}\times$ base × altitude of $\Delta$ ABC

$= \frac{1}{2}\times8\times$ AD

$\therefore\ 6\sqrt{11} = 4\ AD$

$\therefore\ AD = \frac{6\sqrt{11}}{4} = \frac{3}{2}\sqrt{11}\ cm$

$\therefore$ The length of the perpendicular from A to base $\overline{BC} = \frac{3}{2}\sqrt{11}\ cm$

## EXERCISE 14.1

1. Find the area of the equilateral triangle having length of each side 6 units.
2. Find the area of the right angled triangle whose hypotenuse has the length 17 *cm* and has length of its base 15 *cm*.
3. Find the area of the triangle with the length of the sides 36 *cm*, 48 *cm* and 60 *cm*.
4. If the lengths of the sides of a triangle are in proportion 3 : 4 : 5 and the perimeter of the triangle is 120 metre, then find the area of the triangle.
5. An isosceles triangle has perimeter 30 *cm* and length of its congruent sides is 12 *cm*. Find the area of the triangle.
6. The triangular side walls of a flyover have been used for advertisements. The sides of the walls have lengths 100*m*, 35*m* and 105*m*. The rent per year for the advertisements is ₹ 4000 per $m^2$. A company hired one of its walls for 2 months. How much rent did it pay ? ($\sqrt{34} \cong 5.83$)
7. Find the area of the triangle with the lengths of the sides 5 *cm*, 7 *cm* and 10 *cm*. Also find the length of the altitude drawn from the vertex to the side whose length is 10 *cm*.

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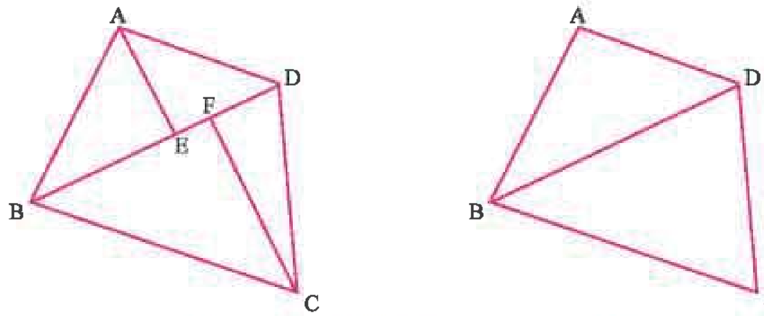
## 14.3 Application of Heron's Formula in Finding Area of Quadrilaterals

For a quadrilateral ABCD, if we join two opposite vertices, then we get a diagonal and if we draw the perpendiculars from remaining two vertices to the diagonals, then we have a formula to find the area of the quatrilateral as

**Area of the quadrilateral = $\frac{1}{2}$** ***(length of a diagonal) (sum of the length of perpendiculars drawn to the diagonal from other two vertices)***

But it is a difficult and tedious process. So instead of it, if we draw a diagonal then quadrilateral region can be divided into two triangular regions and then we can use the fact that area of the quadrilateral = sum of the areas of both triangles. Both these cases are shown in the figure 14.7.




(i) Figure 14.7 (ii)

In figure 14.7 (i) we have the diagonal $\overline{BD}$ and the altitudes are $\overline{AE}$ and $\overline{CF}$. So by finding their lengths (i.e. AE and CF) we can use the result. In figure 14.7 (ii) by a single diagonal we get two triangles and by Heron's formula we can find the area of both the triangles and then take the sum of them. Thus we get the area of the quadrilateral. It will be easier to find the area of a quadrilateral in this manner.

Let us understand this discussion by the following examples.

**Example 4 :** In quadrilateral ABCD, AB = 3 *cm*, BC = 4 *cm*, CD = 6 *cm* and DA = 5 *cm* and the length of the diagonal $\overline{AC}$ is 5 *cm*. Find the area of □ABCD.

**Solution :** Here diagonal $\overline{AC}$ partitions □* ABCD in two triangular regions : Δ*ACD and Δ*ABC. For ΔACD,

$s = \frac{AD+DC+AC}{2} = \frac{5+6+5}{2} = 8\ cm$

Now the area of ΔACD $= \sqrt{8(8-5)(8-6)(8-5)}$

$= \sqrt{8(3)(2)(3)}$

$= 12\ cm^2$

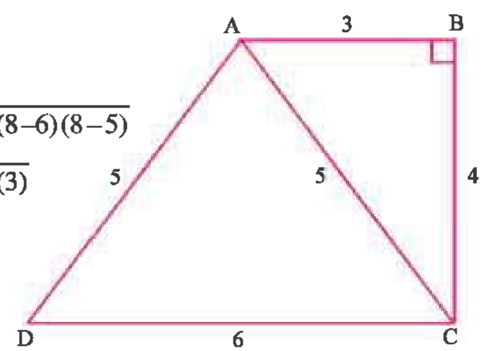



Figure 14.8

For ΔABC, $s = \frac{AB + BC + AC}{2}$

$= \frac{3+4+5}{2} = 6\ cm$

Now the area of ΔABC $= \sqrt{6(6-3)(6-4)(6-5)}$

$= \sqrt{6(3)(2)(1)} = 6\ cm^2$

$\therefore$ Area of $\square$ ABCD = Area of $\Delta$ ACD + Area of $\Delta$ ABC

$= 12 + 6$

$= 18\ cm^2$

See that $\Delta$ABC is a right angled triangle. $\Delta$ADC is an isosceles triangle. So there is no need to use of Heron's formula. Do it by yourself.

**Example 5 :** A park is in the shape of a quadrilateral ABCD, where $m\angle C = 90$. Lengths of the sides are AB = 11 *m*; BC = 3 *m*, CD = 4 *m*, AD = 8 *m*. Then find the area of the park.

**Solution :** Here, for the quadrilateral ABCD, $m\angle C = 90$, and $\overline{BD}$ = diagonal. (figure 14.9). Thus for right angled $\Delta$ BCD, see that we $\overline{BD}$ is the hypotenuse.

$\therefore BD^2 = CD^2 + BC^2 = (4)^2 + (3)^2 = 25$

$\therefore$ BD = 5 = length of the diagonal

Now the area of quadrilateral ABCD

= The area of $\Delta$ BCD + The area of $\Delta$ ABD

$\therefore$ The area of $\Delta$ BCD

$= \frac{1}{2} \times \text{base} \times \text{altitude}$

$= \frac{1}{2} \times BC \times CD$

$= \frac{1}{2} \times 3 \times 4$

$= 6\ m^2$

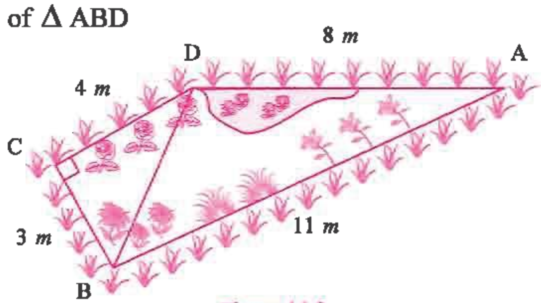


Figure 14.9

Now, for the area of $\Delta$ ABD,

$$s = \frac{AB+BD+AD}{2} = \frac{11+5+8}{2} = 12\ m$$

$\therefore$ Area of $\Delta$ ABD $= \sqrt{12(12-5)(12-8)(12-11)}$

$= \sqrt{12 \times 7 \times 4 \times 1}$

$= \sqrt{4 \times 3 \times 7 \times 4}$

$= 4\sqrt{21}\ m^2$

$\therefore$ Area of quadrilateral ABCD $= 6 + 4\sqrt{21}\ m^2$

## EXERCISE 14.2

1. Find the area of the quadrilateral ABCD where AB = 7 *cm*, BC = 6 *cm*, CD = 12 *cm* and AD = 15 *cm* and the length of the diagonal $\overline{AC}$ is 11 *cm*.
2. Find the area of the quadrilateral ABCD where AB = 8 *m*, BC = 15 *m* and CD = 13 *m*, DA = 12 *m*, $m\angle B = 90$.

3. If the perimeter of a quadrilateral ABCD is 92 *m* and the perimeter of $\Delta$ ABD is 90 *m*, then find the length of the diagonal $\overline{BD}$. Also find the area of the quadrilateral ABCD where AB = 40 *m*, BC = 15 *m*, CD = 28 *m*, DA = 9 *m*.
4. If the lengths of the diagonals of a quadrilateral field are 40 *m* and 24 *m* and they bisect each other at right angles, then find its area.
5. If the lengths of the sides of a parallelogram are 13 *cm* and 10 *cm* and the length of one of its diagonal is 9 *cm*, then find its area.

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## EXERCISE 14

1. Find the area of regular hexagon ABCDEF (figure 14.10) where the length of each side is 4 *cm* and O is the midpoint of the diagonals $\overline{FC}$, $\overline{DA}$ and $\overline{BE}$ and their lengths are 8 *cm*.

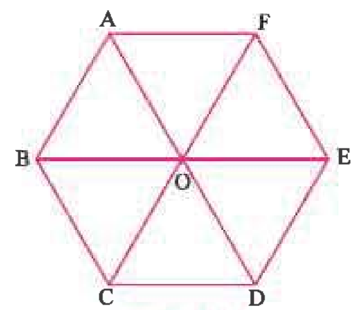

Figure 14.10

2. Find the area of the quadrilateral ABCD, where AB = 9 *cm*, BC = 10 *cm*, CD = 12 *cm*, DA = 11 *cm* and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
3. A bulk of triangular tiles of the length 3 *cm*, 4 *cm* and 5 *cm* is to be used for the flooring of a room with area 216 $cm^2$. Find how many tiles should be used for the flooring. Find the total cost of polishing the tiles at the rate of ₹ 2.75 per $cm^2$.
4. An umbrella is to be made by stitching 8 triangular pieces of cloth with lengths 17 *cm*, 17 *cm* and 16 *cm*. Find how much cloth is required for the umbrella.
5. Find the area of the triangle whose length of the sides are 6 *cm*, 8 *cm* and 10 *cm*.
6. If the length of the sides of a triangle are in proportion 25 : 17 : 12 and its perimeter is 540 *m*, then find the lengths of the largest and smallest altitudes.
7. In figure 14.11, BC = 5 *cm*, CD = 3 *cm*, CF = 6 *cm*. Find the area occupied by the prism on the prism table.
8. The base of a triangular field is twice to its altitude and the cost of cultivating the field is ₹ 30 per hectre and the total cost is ₹ 480. Find the length of the base and altitude of that trianguler field. (10000 $m^2$ = 1 Hector)

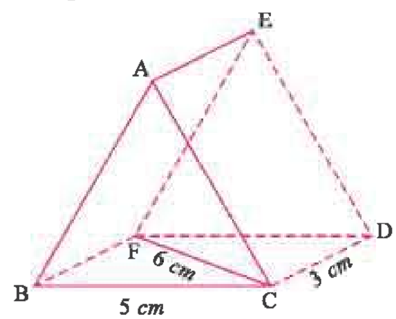

Figure 14.11

9. If the length of the side of a square is 5 *m* and it is converted into a rhombus whose major diagonal has length 8 *m*, then, find the length of the other diagonal and also find the area of the rhombus.

**10.** If the area of a rhombus is 100 $cm^2$ and the length of one of its digonal is 8 *cm*, then find the length of the other diagonal.

**11.** Both of the parallel sides of a trapezium are 8 *cm* and 16 *cm*. Non-parallel sides are congruent, each being 10 *cm*. Then find the area of the trapezium

**12.** Select proper option (a), (b), (c) or (d) and write in the box given on the right so that the statement becomes correct :

(1) For the $\Delta$ ABC, semiperimeter is ...... where AB = 8 *cm*, BC = 6 *cm*, AC = 10 *cm*. ☐

(a) 24 (b) 20 (c) 12 (d) 16

(2) For a $\square^m$ ABCD, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$. If AB = 8 *cm* and BC = 10 *cm* the perimeter of the $\square^m$ ABCD is ...... *cm* ☐

(a) 18 (b) 20 (c) 36 (d) 56

(3) If the perimeter of a trapezium is 50 *cm* and the lengths of non-parallel sides are equal to 12 *cm,* then the sum of parallel sides is ...... . ☐

(a) 13 *cm* (b) 26 *cm* (c) 28 *cm* (d) 30 *cm*

(4) If the area of a rhombus is 54 $cm^2$ and the lengths of one of its diagonal is 9 *cm*, then the length of its other diagonal is ...... *cm*. ☐

(a) 9 (b) 12 (c) 27 (d) 90

(5) If the lengths of the sides of a triangle are in proportion 3 : 4 : 5 then the area of the triangle is ...... sq units where perimeter of the triangle is 144. ☐

(a) 64 (b) 364 (c) 564 (d) 864

(6) If the base of an isosceles triangle has length 10 *cm* and its perimeter is 28 *cm,* then the length of each congruent side is ...... *cm*. ☐

(a) 38 (b) 18 (c) 9 (d) 19

(7) If the lengths of the sides of a triangle are 8 *cm*, 11 *cm* and 13 *cm*, then area of the triangle is ....... $(cm)^2$. ☐

(a) 44 (b) 43 (c) 42.82 (d) $8\sqrt{30}$

(8) If the length of the base of a triangle is 12 *cm* and the length of the altitude to that base is 8 *cm,* then the area of the triangle is ...... $(cm)^2$. ☐

(a) 12 (b) 24 (c) 36 (d) 48

(9) If the area of an equilateral triangle is $2\sqrt{3}$ $cm^2$, then the length of each side of the triangle is ...... *cm*. ☐

(a) $\sqrt{2}$ (b) $2\sqrt{3}$ (c) $2\sqrt{2}$ (d) $3\sqrt{2}$

(10) In a $\Delta$ ABC, $\overline{CD}$ is the altitude of $\Delta$ ABC where AD = 4 *cm*, CD = 5 *cm* and BD = 5 *cm*. Also the area of a square is the same as the area of $\Delta$ ABC. Then length of each side of the square is ...... *cm*. ☐

(a) $\frac{3\sqrt{2}}{5}$ (b) $\frac{3}{2}$ (c) $\frac{3\sqrt{10}}{2}$ (d) $\frac{3\sqrt{5}}{2}$

(11) In a square ABCD, length of each side is 7 *cm*. Then length of its diagonal is ...... *cm* ☐

(a) $\sqrt{2}$ (b) 7 (c) $7\sqrt{2}$ (d) $2\sqrt{7}$

(12) In quadrilateral ABCD, the lengths of each side is shown in the figure 14.12 then the length of the diagonal $\overline{AC}$ is ...... *m*. ☐

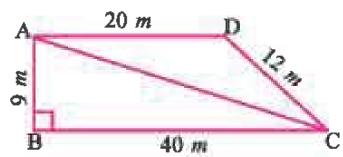


Figure 14.12

(a) 40 (b) 9 (c) 49 (d) 41

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## Summary

In this chapter we have studied the following points :

1. If the lengths of the sides of a triangle are *a*, *b* and *c*, then the perimeter of $\Delta$ ABC is $a + b + c = 2s$ and its semiperimeter is $s = \frac{a+b+c}{2}$.
2. The area of a triangle is given by Heron's formula and it is $\sqrt{s(s-a)(s-b)(s-c)}$.
3. To find the area of a quadrilateral whose sides and one diagonal are given. By a diagonal the quadrilateral region is partitioned into two triangular regions and then by Heron's formula we can find the area of each of the triangles. The sum of areas of both triangles gives us the area of quadrilateral.